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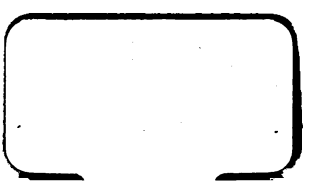

CLASS LESSONS ON EUCLID

CLASS LESSONS ON EUCLID

PART I.

CONTAINING THE

FIRST TWO BOOKS OF THE ELEMENTS

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LONDON

KEGAN PAUL, TRENCH, & CO., 1 PATERNOSTER SQUARE

1882

183. g. 141.

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PREFACE.

THE aim of this little book is to make the utmost use of geometry as a mental training, by leading the beginner to enter into the spirit of Euclid's methods, often rendered obscure by his formal language, and by affording as much assistance as possible in applying them. The difficulty most frequently felt by learners is the application of knowledge acquired to original work, as in the solution of problems. The failure in this important respect is continually adverted to by examiners in their reports and by writers on the subject. May not one of the chief reasons of this failure be found in the mode in which the science is presented to the beginner? In a treatise on algebra or arithmetic, the reasons for the various operations are carefully explained; the student is not expected to infer them from a collection of algebraical or arithmetical examples. On the contrary, in a treatise on geometry, the analytical processes by which the constructions are arrived at are invariably suppressed; the beginner is naturally unable to supply them; consequently he has no principles to guide him in attempting riders, fails in them again and again, and, in too many cases, conceives a distaste for a most valuable and what may be made a most interesting study.

In the following lessons an attempt has been made to

supply this omission, each proposition of Euclid, Books I. and II., being treated as a geometrical exercise proposed for the investigation of the learner, the reasoning which leads to the solution of the problem, or the demonstration of the theorem, is explained in simple language, with the two-fold object of making the propositions *per se* more interesting and intelligible, and of affording models for the guidance of the student in his original work.

After this analysis a summary is made of the main points on which the solution or demonstration depends. These outlines will, it is hoped, be found especially useful in impressing the argument clearly on the mind, and will also effect a great saving of time in a second reading of the subject for examination.

Lastly, each proposition is formally stated succinctly, but avoiding the suppression of any link in the chain of argument. Euclid's demonstration is in all cases given, but a second proof is added where it seemed to present advantages (as in I. 5, &c., and in several propositions of Book II.). The exercises are unusually numerous, and have been principally selected from the last twenty years examination papers of the Universities of London and Cambridge.

The method of teaching geometry indicated above has been adopted by many teachers in their oral lessons; it is hoped that this little work may be found a useful accompaniment to such lessons in the hands of the pupils, and that it may induce other teachers to adopt a plan which is usually attended with excellent results.

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CLASS LESSONS ON EUCLID.

LECTURE I.

THE RISE AND PROGRESS OF THE STUDY OF GEOMETRY.—LIFE OF EUCLID.—REASONS FOR LEARNING GEOMETRY.

THE subject which we are about to begin together may be defined as the science of measurement. It treats of the relations and properties of magnitudes, that is, of those properties of bodies with which we are all familiar under the names of length, breadth or width, and thickness, depth or height.

Geometry, the name of this science, is a word of Greek origin, and means literally land-measuring, thus indicating that men's attention was first drawn to it from the necessary measurements of portions of surface of the ground, for divisions of property, building, and so forth.

Geometry may be divided into two parts, plane and solid. Plane geometry, which we are now going to study, deals with surfaces, that is with figures having length and breadth only.

The Greeks, who may be considered as the educators and thinkers of the old world, just as the Romans were its lawmakers, early paid attention to geometry, and as soon as the science was sufficiently developed made it an important part of their systems of education. From Greece its study spread to other countries, either to the conquered or, as time rolled on, to the conquering races. And from the times of ancient

Greece to our own day, even through the darkness of the early middle ages, geometry has always kept its place as an important part of a liberal education in every civilised country.

The text-book of geometry which we are going to study was written by a Greek philosopher more than 2000 years ago, and hence, next to the Old Testament, is the oldest book in the world in everyday use. Eucleides—or, as we call him in English, Euclid—was a mathematician and musician who lived and taught in Alexandria in the third century before the birth of our Lord. This century is a very remarkable era in the history of Greek literature and art. Euclid lived in the same age as Plato and Aristotle, the founders of the two most celebrated systems of philosophy; as the famous orator Demosthenes; as Praxiteles the sculptor, and Apelles the great painter of antiquity. He dwelt, too, in the highly civilised and very learned city of Alexandria, the celebrated library of which was already in existence. Ptolemy Lagus and his son Philadelphus, who then ruled in Egypt, encouraged learned men at their Court, and the second Ptolemy himself attended the lectures of Euclid.

Euclid founded the famous mathematical school of Alexandria, to which all mathematicians repaired till the taking of the city by the Saracens in the middle of the seventh century. The work in which he embodied his mathematical teaching is called 'The Elements of Euclid.' It is divided into thirteen books, containing the principles of geometry laid down by Pythagoras, Thales, and other philosophers, collected, systematically arranged, and further developed. To these thirteen books two others are generally added, which are now supposed to have been written by Hypsicles, another mathematician of the Alexandrian school. The first six books of the 'Elements' relate entirely to plane geometry, and have been universally studied from the date of their composition down to the present time; the next four are upon arithmetic and a part of the science of numbers which is now generally included in algebra, and are quite superseded by modern writers; the remaining three books deal with the subject of

solid geometry. Parts of the eleventh and twelfth books are still read in our universities.

During the dark ages 'The Elements of Euclid' were not studied in Europe. The geometry then taught consisted merely of a few definitions and axioms, without any attempt at mathematical demonstrations. The Saracen conquerors of Alexandria, however, became familiar with the 'Elements,' and translated it into Arabic, and from Arabic manuscripts what we may consider as the first modern edition of Euclid was translated by Adelard of Bath, an English traveller, who, at some period of the twelfth century, visited the schools of Cordova and Bagdad. The first printed 'Euclid,' which was published at Venice in 1482, was also a translation from the Arabic. The first translation from Greek manuscripts was in 1505. These were all translations into the Latin language, which was throughout the middle ages the universal language of scholars. In Queen Elizabeth's reign Billingsley, a citizen of London, translated the 'Elements' into English, and from his time to our own numerous editions and commentaries have continually issued from the press, while, at the same time, from the revival of letters to the present day, Euclid's work has been studied in the schools and universities of Europe.

Next, before we begin to study the book whose history we have briefly surveyed, let us, in order that we may pursue our subject intelligently, consider what ends we have in view in learning this science of geometry. The advantages are, I think, twofold—direct and indirect. First, geometry teaches us, as we have already seen, the laws which regulate the relations of the parts to the whole in simple figures, and of these figures to each other. Now, if we look around us, all the objects which meet our glance are either modifications of, or combinations of the simple figures, and having studied the proportions of the elementary figures in the abstract we shall be able to apply them to all the surrounding objects of life, and we shall know the fundamental principles upon which painting, architecture, and engineering depend; also on these same

principles depend the useful and the beautiful in furnishing and adorning our houses, in laying out our gardens, and even in designing our dress.

Again, the study of geometry tends to develop and cultivate some of our most valuable mental qualities. By it the power of reasoning is continually exercised. Euclid does not call upon us to believe or to take on authority anything but a very few self-evident truths. Every assertion that he makes he afterwards proves. And, in order to comprehend his reasoning, we must be careful to follow the argument closely, or, by a moment's inattention, a point will escape our notice, and we shall find at the conclusion a flaw in the demonstration which will make it valueless, and oblige us to retrace our steps to the very commencement. This power of fixing the attention, or of concentrating our ideas, is most necessary to our ever succeeding in attaining any great or worthy object.

We have now traced the origin of the science of geometry, and the history of its development, chiefly under the influence of Euclid's genius. We have considered, also, some of the reasons for studying it. We shall, in our next lesson, proceed to the consideration of the first book of the 'Elements.'

LECTURE II.

ON THE DEFINITIONS, AXIOMS, AND POSTULATES.

Our lesson to-day will embrace (1) the definitions in which Euclid gives the precise meaning of the terms he intends to employ; (2) the postulates in which he asks to be allowed to do three simple things, and to take for granted three truths belonging to geometry alone; and (3) the axioms, a list of nine truths, which, by the nature of our minds, we perceive intuitively, and which are common to all magnitudes to which they can be applied.

First, the definitions, 35 in number (in the original ὅροι, boundaries); these, in order to assist our memory, we may classify as follows:—

Nos. 1–7 are definitions of lines and surfaces.
„ 8–12 „ of angles.
„ 13–34 „ of figures.

Of these

Nos. 15–19 define the circle and its parts.
„ 24–29 „ triangles.
„ 30–34 „ four-sided figures.
No. 35 defines parallel straight lines.

OF LINES AND SURFACES.

1. 'A point is that which has no parts or magnitude.'

2. 'A line is length without breadth.'

3. 'The extremities of a line are points.'

4. 'A straight line is that which lies evenly between its extreme points.'

5. 'A superficies or surface is that which has only length and breadth.'

6. 'The extremities of a superficies are lines.'

7. 'A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.' (This is the carpenter's test, trying a flat surface with a straight edge.)

The ideas which are formulated in these definitions have been familiar to us, though perhaps unconsciously, from early childhood. The uncut end of the pencil I hold in my hand, the outside of the cover of the book on the table, or any of its pages, are plane surfaces (def. 7). The polished wood of the pencil forms a rounded or curved surface; both kinds of surface are included in def. 5. Again, the edges of the book are lines (def. 6) having length without breadth (def. 2). Any two edges of the book meet in a point (def. 3) which obviously has no size or magnitude (def. 1).

To these definitions an objection is frequently urged which we must briefly examine. It is said that defs. 1, 2, and 5 are incorrect, since any surface must have some thickness, any line some breadth, and any point some size, however small. We have seen that the outside of the book-cover is a plane surface. Now, suppose we wish to measure this surface in order to find out what space on the table it would occupy; you will tell me that the book is so many inches broad (say 6 in.), and so many inches long (say 9 in.), and that it will therefore occupy a space of so many (say 54) square inches. Now, if I ask, but how thick a space will it occupy? will you not be inclined to smile at such a question and answer, 'The surface of the book contains 54 sq. in., and will therefore cover 54 sq. in. of the surface of the table, but it has nothing to do with thickness, either that of the book or that of the table.' Hence, it is evident that we have a clear conception of surface as possessing the properties of length and breadth without thickness, and that Euclid's definition is strictly true.

Again, if I ask you to measure an edge of the book (which we took as our example of a line), you will tell me it is so many inches long; but if I ask how wide it is, the ques-

tion strikes you as absurd, it has no width, nor could you measure the point at which any two edges meet at all, and it consequently cannot be said to have any size whatever. We must then admit that these definitions give perfectly accurate descriptions of the ideas which we have in our minds, and express by the words surface, line, and point. In other words, we may say that a point indicates position; a line, distance from one point to another; a surface, the extension of any body in one plane, *i.e.* the amount of space it would occupy in one plane surface or area.

But I wish you also to notice that the lines and surfaces with which we are familiar from experience exist always in connection with each other, and also with what we have seen is the third geometrical property of matter, viz. thickness or depth, *i.e.* the lines are boundaries of some surface, the surfaces of some solid body. But in that part of the subject which we are now beginning Euclid requires us to think of surfaces and lines as if they existed separately, just as in comparing colours we should speak of white and red, without any reference to the thing which was white or red. The marks made by a pen or pencil to represent lines are very thin and narrow solids of ink or black lead.

OF ANGLES.

We pass on to the simplest combination of lines, *i.e.* two lines meeting in a point forming a corner or angle (Lat. *angulus*, a corner).

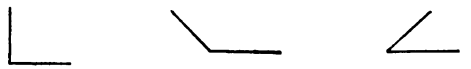
Def. 8. 'A plane angle is the inclination of two lines to one another in a plane, which meet together but are not in the same direction.'

9. 'A plane *rectilineal* angle is the inclination of two *straight* lines to one another, which meet together but are not in the same straight line.'

Def. 8 includes angles formed by two curved lines, or a straight and a curved line. With these we shall not be concerned. The word rectilineal simply

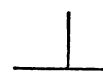
means straight-lined or right-lined (Lat. *rectus*, right; *linea*, a line). In many modern books on geometry a straight line is termed a 'right line.'

No. 9 gives a general definition of the three kinds of plane



angles, right angles, angles greater than a right angle, called obtuse (Lat. *obtusus*, blunted) angles, smaller than a right angle, called acute (Lat. *acutus*, sharp). These are thus defined by Euclid:—

10. 'When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.'



11. 'An obtuse angle is that which is greater than a right angle.'

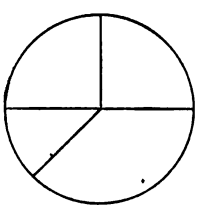
12. 'An acute angle is that which is less than a right angle.'

OF FIGURES.

13. 'A term or boundary is the extremity of anything.'

14. 'A figure is that which is enclosed by one or more boundaries.'

15. 'A circle is a plane figure contained by one line called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal.' These lines are called radii (Lat. *radius*, the spoke of a wheel, which each resembles).



16. 'And this point is called the centre of the circle.'

17. 'A diameter of a circle is a straight line drawn through the centre and terminated both ways by the circumference' (equal in length to two radii).

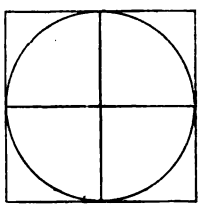
18. 'A semicircle is the figure contained by a diameter, and that part of the circumference (called from its shape an arc, Lat. *arcus*), which it cuts off.'

19. 'A segment of a circle is the figure contained by a straight line (called a chord) and the circumference which it cuts off.'

A semicircle is of course a segment of a circle.

A sector, or section, of a circle is the figure contained by two radii and the arc cut off.

The circumference of a circle is nearly $3\frac{1}{7}$ times (3·14159) the diameter. The area of a circle may be found by multiplying the square of the radius by $3\frac{1}{7}$, the relation of the diameter to the circumference ($\pi\rho^2$), *e.g.* supposing the radius of a circle to be 6 in. in length, its area would be $(36 \times 3\frac{1}{7})$ sq. in. Neither of these statements can be fully proved without some knowledge of mathematics, but their truth may be roughly ascertained, in the first instance by actual measurement, and in the second thus:—Draw a circle of any size; draw it in two diameters cutting each other at right angles. The two diameters cutting at the centre will be divided into four radii, on each of these draw a square, the resulting figure will be equal to four times the square of the radius, and we shall see that the circle does not quite fill it up; hence we should expect that the area of the circle would be equal to rather more than three times the square of the radius.



The circumference of a circle is divided by geometers into 360 equal parts, called degrees. This division will be familiar to most of you from your knowledge of geography; the circles which we imagine to pass round the earth's surface, forming the parallels of latitude and longitude, are divided, as you know, into 360°. If you will refer a moment to the figure above you will see that the four right angles, formed by the diameters cutting each other, each intercept or cut off a quarter of the circumference of the circle, containing, as you

know, 90°. A right angle is therefore often spoken of as an angle of 90°, and angles as well as circles are measured by degrees according to the size of the arc which they intercept. An acute angle being less than a right angle may contain any number of degrees less than 90; an obtuse angle, being greater than a right angle, may contain any number of degrees between 90 and 180.

OF RECTILINEAL FIGURES.

You will readily see that three straight lines are the smallest number that will contain a figure, and also that a figure contained by straight lines will have as many angles as sides; hence these figures may be named either according to their angles or their sides. Thus Euclid speaks of trilateral, three-sided, quadrilateral, four-sided, multilateral, many-sided, and equilateral, equal-sided, figures. Again we have triangle, a three-cornered or three-angled figure, polygon, a many-angled figure (Gr. πολὺς, many, γωνία, an angle), also pentagon, a five-angled figure, hexagon, octagon, &c.

20. 'Rectilineal figures are those which are contained by straight lines.'

21. 'Trilateral figures, or triangles, are contained by three straight lines.'

22. 'Quadrilateral figures by four straight lines.'

23. 'Multilateral figures, or polygons, by more than four straight lines.'

A *regular* polygon has all its sides equal. The perimeter of any figure is the measurement round about it, *i.e.* the sum of its sides.

OF THREE-SIDED FIGURES.

Euclid classifies triangles (1) according to their sides, as equilateral or equal sided, isosceles (ἴσος, equal, σκέλος, leg), equal legged, having two equal sides, and scalene, limping (Gr. σκαληνός), having three unequal sides; (2) according to their angles, as right-angled, obtuse-angled, and acute-angled. It will be shown further on that every triangle contains two

acute angles; consequently a right-angled triangle has one right angle, an obtuse-angled triangle an obtuse angle, and an acute-angled triangle three acute angles.

24. 'An equilateral triangle is that which has three equal sides.'

25. 'An isosceles triangle is that which has two sides equal.'

26. 'A scalene triangle is that which has three unequal sides.'

27. 'A right-angled triangle is that which has a right angle.'

28. 'An obtuse-angled triangle is that which has an obtuse angle.'

29. 'An acute-angled triangle is that which has three acute angles.'

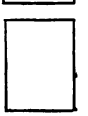
OF FOUR-SIDED FIGURES.

Four-sided figures are of two classes, those having their opposite sides parallel, called parallelograms, and irregular figures, called trapeziums (tables).

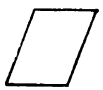
The line joining the opposite angles of any quadrilateral is called a diameter, or diagonal. Parallelograms are of four kinds, which are thus defined by Euclid :—

30. 'A square is that which has all its sides equal and all its angles right angles.'

31. 'An oblong, or rectangle, is that which has all its angles right angles, but not all its sides equal.'



32. 'A rhombus is that which has all its sides equal, but all its angles are not right angles.'



33. 'A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal nor its angles right angles.'

34. 'All other four-sided figures besides these are called trapeziums.'

OF PARALLELS.

35. 'Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

NOTE.—The parts of the circle enumerated in defs. 17, 18, 19, are not referred to afterwards in Book I., and the figures defined in Nos. 31, 32, 33, are spoken of generally as parallelograms.

THE POSTULATES.

The postulates, in the original *αἰτήματα*, requests or petitions, as the word is translated in the old editions of Euclid, come next. In the first three Euclid asks:—

1. 'That a straight line may be drawn from any one point to any other.'

2. 'That a terminated straight line may be produced to any length in a straight line.'

3. 'That a circle may be described from any centre at any distance from that centre.'

Postulates 4 and 5 refer to simple geometrical truths which are quite evident; postulate 6 is not so evident and will be discussed later on—it is inserted here to make the list complete. These three are in many editions placed among the axioms.

4. 'Let it be granted that all right angles are equal to one another.'

5. 'That two straight lines cannot enclose a space.'

6. 'That if a straight line meet two others, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are together less than two right angles.'

THE AXIOMS.

We next have a list of nine axioms, in the original κοιναὶ ἔννοιαι, common notions, statements of facts which we cannot help knowing, but of which we do sometimes require to be reminded.

1. 'Things which are equal to the same thing are equal to one another.'

From this naturally follow axioms 6 and 7.

6. 'Things which are double of the same thing are equal to one another.'

7. 'Things which are halves of the same thing are equal to one another.'

These axioms may of course be extended to any multiple or fraction.

Axioms 2–5 form a second connected group.

2. 'If equals be added to equals the wholes are equal.'

3. 'If equals be taken from equals the remainders are equal.'

If A=C and B=D, $A \pm B = C \pm D$.

4. 'If equals and unequals be added together, the wholes are unequal.'

5. 'If equals be taken from unequals the remainders are unequal.'

If A=C but B and D represent unequal magnitudes, then $A \pm B$ is not equal to $C \pm D$.

8. 'Magnitudes which coincide or exactly fill the same space are equal.'

'Things are said to agree together or coincide if when they are applied the one to the other, or set the one upon the other,

the one exceedeth not the other in any thing.' A magnitude is that which can be measured. The magnitudes with which we are at present concerned are lines having length-measure, plane figures or superficies having the measures of length and breadth or *area*, and angles which, as we have seen, are measured by the size of the arc intercepted. This axiom applies to all these magnitudes as well as to others which are not geometrical.

9. 'The whole is greater than its part.'

LECTURE III.

ON THE NATURE AND ARRANGEMENT OF A PROPOSITION.—PROPOSITIONS 1–3.—RESULTS OBTAINED.

THE subjects treated of in the first book of Euclid are :—

(1) The properties of triangles.

(2) The theory of parallel straight lines, with the properties of parallelograms.

(3) The laws which determine equivalence or equality of area in rectilineal figures.

These subjects Euclid discusses in forty-eight propositions (*πρότασεις*), which are of two kinds, (1) problems in which he shows how something may be done, *e.g.* how to describe an equilateral triangle, and (2) theorems in which he proves some geometrical law, *e.g.* the angles at the base of an isosceles triangle are equal.

A proposition consists of the enunciation, the construction and the proof.

The enunciation states in general terms the problem to be solved or the truth to be demonstrated, *e.g.* the angles at the base of an isosceles triangle are equal, and (2) applies this general statement to a particular instance; thus let ABC be an isosceles triangle having the side AB equal to the side AC, then the angle ABC shall be equal to the angle ACB. These divisions are sometimes distinguished as the general and particular enunciation.

The construction describes the lines and circles, &c., which are to be drawn in order to solve the problem or demonstrate the theorem. Sometimes, however, no construction is required.

The proof shows by reasoning that the problem proposed has been solved, or that the theorem is true.

There are two methods of proof, 'direct' and 'indirect.' A direct proof shows the truth of a proposition directly by means of other truths already established. An indirect proof shows the truth of a proposition negatively, by proving that the truth of any supposition contrary to that in the enunciation being granted, we arrive at untrue or absurd results.

The various lines and angles, &c., made use of in a proposition are named from the letters of the alphabet taken consecutively. A line is usually named by two letters placed at its extremities, thus the line AB *a*——*b*, an angle by three, placing the letter at the angular point in the middle, thus the angle ABC or CBA. A rectilineal figure usually has a letter placed at each angle.

There are a few contractions in general use in geometry.

Q.E.F. problems are usually concluded by these letters, an abbreviation for 'quod erat faciendum,' which was to be done.

Q.E.D. stands for 'quod erat demonstrandum,' which was to be demonstrated, and forms the conclusion of theorems.

Hyp=Hypothesis. This is used in referring to the supposition involved in the enunciation, *e.g.* the angles at the base of an isosceles triangle are equal. In this proposition we consider two sides of the triangle to be equal, in accordance with the supposition of an isosceles triangle in the enunciation, that is by hypothesis.

The following symbols greatly shorten the labour of writing out demonstrations:—

= is equal to	$\triangle$ triangle
$\angle$ angle	sq. square
rt. $\angle$ right angle	$\therefore$ therefore
$\parallel^{gram}$ parallelogram	$\because$ since or because

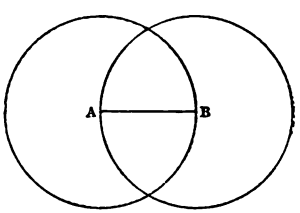
PROP. I. Having now cleared the way, we can enter upon the discussion of the first proposition, which is a problem, that is, it shows how something is to be done. We are

required 'to describe an equilateral triangle on a given finite straight line.' 'Describe' is a technical term denoting to draw or construct a mathematical figure.

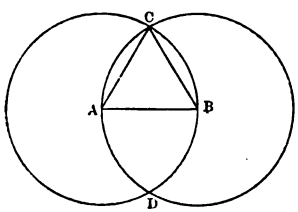
Def. 24 tells us that an equilateral triangle is a figure having three equal sides.

The given line will itself form one side of the triangle, and if we can draw two other lines equal to it in such a position that the three lines shall together form an equilateral triangle, the problem will be solved.

Draw a line to represent the given line A_____B. As the line AB is to form one side of the triangle, its extremities will be the points from which the other two sides must be drawn. We will endeavour to obtain these by the help of the principles already learnt. Of these the postulates are the most likely to be of assistance, since they alone permit us to do anything. We will examine them in order. The first postulate assumes that we may draw a 'straight line from any one point to any other.' Will this help us here? No; it affords no means of measuring the length of the lines which we have to draw. The second postulate, that 'a terminated straight line may be produced to any length,' fails to help us, for the same reason. We have remaining the third postulate, which assumes that 'a circle may be described from any centre at any distance from that centre.' Let us refer to the definition of a circle (def. 15). It is to the latter part that I wish to draw your attention, 'and is such that all straight lines drawn from a certain point (the centre) within the figure to the circumference are equal.' Here is a mode of measuring the lines required. AB will form a radius of a circle, and if we take A as the centre, we can draw as many lines as we please from A to its circumference, which will be equal to AB. In like manner, by drawing a circle, of which B is the centre, with AB again as radius, we shall obtain a measure for lines



equal to AB drawn from the point B. But to form a triangle these lines must meet at a point. How shall we find this point? Looking at the figure, you will see that there are two points at which the circles cut one another, which we will call C and D, and at each of these, two radii, one drawn from the centre of each circle (*i.e.* from A and B), would meet. Take the point C and draw lines AC and BC from points A and B (post. 1). An equilateral triangle is thus described; for



since BC and AC are each equal to AB (def. 15), they are equal to each other (ax. 1).

Thus the two conditions of the enunciation are satisfied, for ABC is an equilateral triangle, and it is described upon the given line AB.

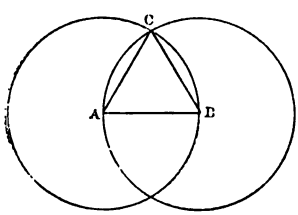
It remains to translate the problem into formal mathematical language, and it will be identical with Euclid's first proposition.

Proposition I., Problem.

[The propositions are collected at the end of the book for convenient reference. It is intended that each should be read immediately after its analysis.]

Summary of Proposition I., Problem 1.

To describe an equilateral triangle on a given straight line AB.



Cons.—With common radius AB, describe circles from centres A and B. These will cut at point C. Join AC and BC. ABC is the △ required.

Proof (Direct).—

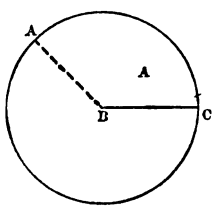
$$\left.\begin{matrix} AC= \\ BC= \end{matrix}\right\} AB \text{ (def. 15).}$$

$\therefore$ BC = AC (ax. 1), and △ ABC is equilateral, &c.

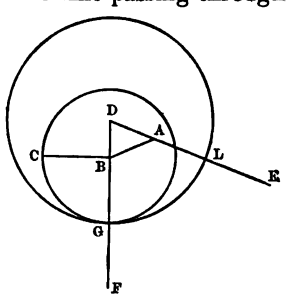
Q.E.D.

Prop. II. We pass on to the second proposition, also a problem, 'From a given point to draw a straight line equal to a given straight line.' Mark any point A and draw a straight line BC to represent the given line. From our work in the last proposition you will remember that we can obtain lines equal to any given line by describing a circle of which it forms a radius, so that by taking BC as radius of a circle we can draw any number of lines equal to it, from the centre of the circle. Draw this circle, taking B as its centre. Now supposing the point A should fall anywhere on its circumference, the problem will be solved by drawing a line AB to the centre, which will form a radius and hence be equal to BC.

But supposing that A does not fall so, but at any point within or without the circumference of the circle, how shall we obtain a measure for any line drawn from the point A? Here lies the difficulty of the problem, which Euclid overcomes by a very ingenious use of Prop. 1. He tells us to draw a straight line from A to B and on it describe an equilateral triangle (I. 1), and to produce its sides DA and DB through the circle already drawn to any points E and F. This gives the key to the solution, which we ought to be able to complete.



The lines DF and DE are each divided into three parts; what do we know about them? Of the line passing through A we know that DA = DB, since they are sides of the equilateral triangle; we know nothing as to the length of the other parts. Looking at DF we have DB = DA, as was said, and also BG = BC, the given line (def. 15), or, adding these parts together, we may say that DG = a side of the triangle + the given line. Now drawing a circle of which DG is radius and D centre, we cut off on line

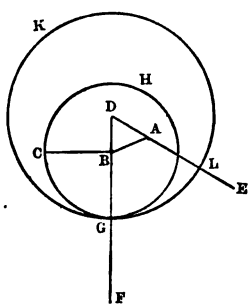


DE a part DL = DG, *i.e.* a side of the triangle + the given line. Take away DA the side of the triangle, the remainder AL must be equal to the given line, and it is therefore the line required.

There are eight different ways in which the figure of this proposition can be drawn, according to the relative position of the point A and BC.

Summary of Proposition II., Problem 2.

From a given point to draw a straight line equal to a given straight line.



Cons.—Join AB, and on it describe the equilateral Δ DAB.

With radius BC draw circle GCH. Produce DB to meet it in G.

With radius DG draw circle GKL, produce DA to meet it in L.

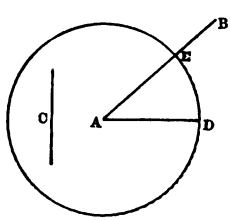
Proof.—

DL = DG (def. 15).

DA = DB (def. 24).

∴ remainder AL = BG = BC (ax. 3, def. 15).

Proposition III. is also a problem, 'From the greater of two straight lines to cut off a part equal to the less.' The given lines may, or may not, be drawn from one point. First, supposing that they are, take any point A, and from it draw two straight lines AB and AD, of which let AD be the shorter. Here we can at once apply our line-measure the circle, for, taking A as centre and the shorter line as radius, a circle can be drawn which will cut AB in E, making a part AE equal to AD (def. 15). Next, supposing that the extremities of the lines do not meet, and that a line C is the shorter. From



the last proposition we have learnt 'to draw a line from any point equal to any given line;' hence from point A a line can be drawn equal to C, and then we can proceed as in the first case.

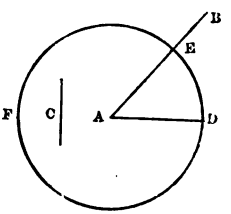
[NOTE.—In the formal statement of this proposition it is not necessary to divide it into two cases, as the first is included in the second. This has been done here only to make the solution more obvious.]

SUMMARY OF PROPOSITION III., PROBLEM 3.

From the greater of two given straight lines, AB and C, to cut off a part equal to the less.

Cons.—Make line AD = C (I. 2).
Draw circle DEF, cutting AB at E.

Proof.—AE = AD = C (def. 15, cons. ax. 1).



This proposition concludes the first group of problems, by which we have learnt to do three things in addition to those assumed in the postulates. We can—

Describe an equilateral triangle (I. 1).

Draw a straight line equal to any other (I. 2).

Cut off from any line a part equal to any lesser line (I. 3).

Throughout these propositions the circle has been made use of as a line-measure.

LECTURE IV.

THE METHOD OF SUPERPOSITION.—THE INDIRECT PROOF.—PROPOSITIONS 4–6.

PROPOSITION IV. is the first of a group of theorems which continues till the ninth. A theorem, as you will remember, demonstrates some geometrical truth. The truth here demonstrated is that: If one triangle has two sides and the angle included by these sides, severally equal to two sides, and the angle included by them of another, these triangles are equal in all respects.

Let us consider in what respects triangles can be said to be equal to one another:—

(1) In their sides; each side of the one must be equal to the corresponding side of the other.

(2) In their angles; each angle of the one must be equal to the corresponding angle of the other.

(3) In their area, or the space enclosed by the sides.

And these three equalities you will find very rigidly defined in Euclid's enunciation of this theorem, which is given below.

In the triangles which we are considering we are given—

(1) Two sides in one equal to two sides in the other, each to each.

(2) One angle (the angle included by the equal sides) in one equal to the corresponding angle in the other.

We are required to show that—

(1) The third sides, or bases, are equal.

(2) That the two remaining angles in each triangle are equal each to each.

(3) That the areas are equal.

Thus, in one demonstration, we have to show the equality of three different kinds of magnitudes, lines, angles, and area or surface. Have we any test of equality which will apply to them all? Axiom 8 tells us that magnitudes which coincide are equal, and, further, it was pointed out in a note that the test of coincidence could be applied to the three magnitudes in question. But how are we to apply this test? From the same note we gather that one triangle must be set upon or applied to the other. We will then imagine triangle ABC to be applied to DEF. Now, since in the two triangles two sides and *the angle between them* are given equal, we know that these equal parts will coincide. What results can we observe from this? First, since the equal sides coincide the bases must coincide, for the extremities of the bases are also the extremities of the sides which coincide, and if the extreme points of two straight lines coincide the whole lines must do so also, for if not two straight lines would enclose a space.

Next, since the bases coincide, all the sides of the triangles coincide. Therefore the parts contained by them must coincide; these are (1) the area, *i.e.* the whole space included by the sides; (2) the two remaining angles in each triangle contained by the base and one of each of the equal sides.

Finally, the parts which coincide must, as we have seen above (ax. 8), be equal.

Therefore the triangles ABC, DEF, are equal as to their sides, their angles, and their area—Which was to be proved.

Summary of Proposition IV., Theorem 1.

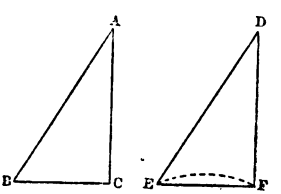
Given: In two triangles two sides and the included angle equal.

To be demonstrated: That the triangles are equal in all respects.

Cons.—Nil.

Proof.—Apply one triangle to the other.

Those parts which are given equal will coincide.



∴ the equal sides coincide at all points, the bases must coincide (post. 5), and are equal (ax. 8).

∴ all the sides coincide, the parts included by them, *i.e.* the areas and remaining ∠s, coincide and are equal (ax. 8).

The method of superposition, as it is called, is here made use of. It consists in supposing one figure to be laid upon another for the purpose of comparison, and depends upon the eighth axiom. It is made use of sparingly by Euclid, but more often by later geometers of the Alexandrian school and by modern writers.

The third sides of the triangles in this proposition are spoken of as 'bases.' This name may be applied to any side of a triangle to distinguish it from the other two, except in an isosceles triangle, when it always refers to the unequal side. The point opposite the base, at which the other two sides of a triangle meet, is called the 'vertex,' and the angle at this point the 'vertical angle.'

Prop. V. Before beginning the next theorem, I wish you to notice that from Proposition IV. we learn (1) that triangles are equal in all respects, if in each triangle two sides and the angle between them are severally equal; *therefore* if we want to show that any two triangles are equal we must prove that two sides and the included angle in one triangle are equal to the corresponding parts in the other. (2) That the triangle affords a means of comparing angles, for by showing the equality of the triangles in the last proposition we were able to prove the equality of two pairs of angles.

You will see that we can include, or shut in, any angle in a triangle, by drawing a straight line meeting the lines containing the angle, or the arms of the angle as they are called.

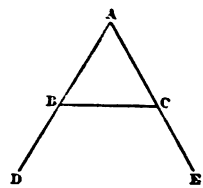
With this preface I hope that we shall not only be able to understand, but to work out for ourselves the famous fifth proposition.

We are required to show:

(1) That the angles at the base of an isosceles triangle are equal.

(2) That if the equal sides are produced the angles on the other side of the base are equal.

Draw an isosceles triangle ABC of which side AB = side AC we have to show (1) that angle ABC = angle ACB.



Produce the equal sides to D and E we have to show (2) that angle DBC is = to angle ECB.

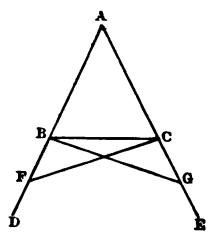
We will first endeavour to show that angle DBC is equal to angle ECB. We have said that in order to compare angles we must include them in triangles. Now look at the figure and you will see that we can construct a triangle containing angle DBC by drawing a line from C to any point F in BD. In the same way angle BCE can be included in a triangle by drawing a line from B to a point G in CE. But in order to show that the angles are equal we must show that the triangles containing them are equal; therefore in constructing the triangles we must do what we can to make them equal, so let us take point G, so that CG (a side of one triangle) is equal to BF a side of the other.

We will pause to consider our figure, we have now :—

(1) The given triangle ABC.

(2) A pair of constructed triangles FBC, GCB.

(3) A pair of triangles AFC and AGB formed by adding the given triangle to each of the constructed ones.



You will see, too, that as the triangles overlap each other, some of the sides and some of the angles are common to two or even three triangles. Thus, taking the angles first, we have the angle at A common to three triangles, the given triangle and the larger pair; the angles at F and G are each common to one of the larger and one of the smaller con-

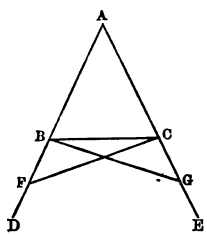
structed triangles. Similarly, taking the sides, we have BC common to three triangles; FC and BG, each of them common to two, as also BA and AC.

Let us proceed to compare the smaller pair of triangles, for if we can show that they are equal, it follows that 'the angles on the other side of the base' are equal.

In the triangles FBC, BCG:

we have side BF = CG (by construction)
side BC = BC (common to both).

We know nothing of the size of any of the angles, but the angles at F and G, as also sides FC and BG, are, as we have said, common to the larger and smaller pair of triangles. Let us then compare the larger triangles, and if they are equal we can retrace our steps and show that the smaller pair are.



In triangles FAC, GAB, we have
side AB = side AC (sides of the isosceles triangle)
side AF = side AG, since their parts are equal (AB = AC and BF = CG *cons.*)

The angle at A is common to both, and is contained by the equal sides, therefore by Proposition IV. these triangles are equal in every respect. We return to the smaller pair of triangles; we can now assert that

side BF = side CG (*cons.*)
side FC = side BG (since they are sides of larger pair of triangles shown to be equal by I. 4),

and the angles at F and G, included by the equal sides, are equal (since they are also angles of larger pair of triangles); these triangles therefore are equal in all respects (I. 4), and angles DBC and ECB, the angles on the other side of the base, are equal.

We must now show that the angles at the base are equal, angle ABC to angle ACB. Glancing at the figure again you will see that the angles ABG and ACF are each made up of

one of the 'angles at the base' and one of the angles of the smaller pair of equal triangles, or, to put it more clearly,

$$\angle ABG = \angle ABC + \angle CBG$$
$$\angle ACF = \angle ACB + \angle BCF.$$

And we know that angles ABG and ACF are equal, since they are similar angles of the large pair of triangles. Also angles CBG and BCF are equal, since they are similar angles of the smaller pair of triangles. Then subtracting the equal parts from the equal wholes we have equal remainders (ax. 3), that is the angle ACB is equal to the angle ABC, and these are the angles at the base.

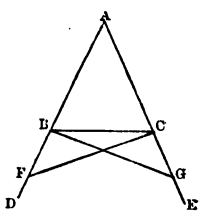
Summary of Proposition V., Theorem 2.

The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles on the other side of the base are equal.

Cons.—In BD take any point F. From CE cut off CG = BF (I. 3), join FC, GB.

Proof.—By I. 4 prove (1) the larger pair of triangles equal, (2) the smaller pair of triangles equal.

Since the smaller Δs contain 'the $\angle$s on the other side of the base,' these angles are equal.



Subtract from the $\angle$s of the larger triangles, the $\angle$s of the smaller which form a part of them, and the remainders, 'the $\angle$s at the base,' are equal (ax. 3).

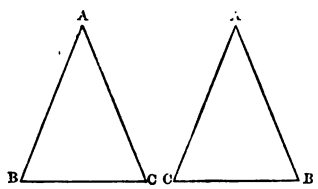
From this proposition it is plain that all the three angles of an equilateral triangle are equal.

A truth which follows immediately from the proof of any theorem is called a 'corollary.'

Proposition V. is of very ancient origin, and is ascribed to the philosopher Thales, one of the seven wise men of Greece.

Another demonstration of its first part is often adopted depending on the method of superposition, which is said to have been invented by Pappus, a geometer of the Alexandrian school who lived in the fourth century.

Imagine the isosceles triangle ABC to be turned over and set down the reverse way. This will give, as it were, the figure of two triangles ABC and ACB.



In these AB = AC

AC = AB.

The angle at A is common to the two.

Therefore the triangles are equal in all respects (I. 4),

and the angle ABC = ACB = ACB.

Proposition VI. is what is called the converse of the first part of Proposition V., *i.e.* we are *given* what we were required to prove before, and we have to prove what we were then given.

e.g. Proposition V. given—Two sides in a triangle equal; show that the two opposite angles are equal.

Proposition VI. given—Two angles equal; show that the two opposite sides are equal.

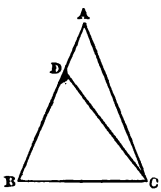
Euclid here, as in many other converse propositions, adopts the indirect method of proof. This, which is not so satisfactory to most minds as the direct proof, consists in showing that an absurdity or an impossibility, such as the contradiction of an established truth, necessarily follows from any supposition contrary to that in the enunciation. We have had an instance of the 'reductio ad absurdum' in I. 4 when it was shown that if the bases of the two triangles did not coincide, two straight lines would enclose a space, but as it was also

made evident by superposition that they did coincide, this proof cannot be considered an example of indirect demonstration.

The method will be best understood by an example, and as the argument is very simple we will take the proposition in Euclid's words. (See page 115.)

Summary of Proposition VI., Theorem 3.

The angles at the base of a triangle being given equal, to show that the sides opposite to them are equal.



Cons.—Assuming AB to be greater than AC, make DB equal to it (I. 3), join DC.

Proof (*Indirect*).—Show that $\triangle$ DBC = $\triangle$ ABC (I. 4).

This is absurd (ax. 9).

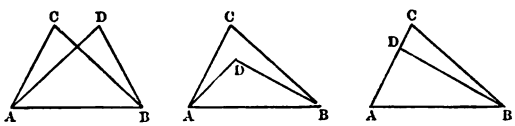
$\therefore$ AB is not unequal to AC, *i.e.* AB = AC.

LECTURE V.

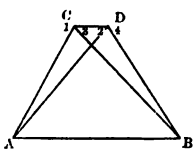
PROPS. 7–8.—RESULTS OBTAINED FROM THE GROUP OF THEOREMS CONCLUDED.

THE enunciation of Prop. VII. is negative, which is very unusual. It states that 'on the same base, and on the same side of it, there *cannot* be two triangles which have their sides which are terminated at one extremity of the base equal and *likewise* those which are terminated at the other.' An example will make the meaning of this clear: Let ABC, ABD be two triangles on the same base AB, if AC = AD it is impossible that BC should = BD; or, if BC = BD, AC cannot = AD.

In drawing this figure you will observe that the triangles can be arranged in three different ways, each representing the conditions of the enunciation. (1) The vertex of each triangle may be outside of the other; (2) or, the vertex of one



triangle may fall inside the other; or (3), it may fall on a side of the other. In the last case it is plainly impossible that AD should be equal to AC, the part to the whole.



We must now consider the first two cases which are proved by the indirect method and by means of the 5th proposition. In accordance with the indirect method we will assume that to be true which we wish to dis-

prove, and endeavour to deduce some absurd result from the assumption. Supposing AC and AD to be equal, and also CB and DB, by joining CD the vertices of the triangles we construct two isosceles triangles, with a common base CD. What do we know further about these triangles? Because they are isosceles the angles at their bases are equal, *i.e.*

Angle (1) ACD = angle (2) ADC.
„ (3) BCD = „ (4) BDC.

Examining these four angles further we observe that two of them are parts of the other two. Hence:—

1 is greater than 3,
4 is greater than 2.

Combining this information, we may infer that—

1 is greater than 4 (since 3 = 4)
and that 4 is greater than 1 (since 2 = 1) } which is absurd,

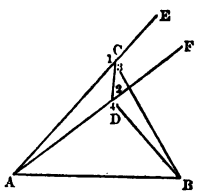
and consequently the lines ending at A, and also those at B, are not equal.

Or, the angles may be compared as follows:—

Since 1 = 2 and is greater than 3, 2 is greater than 3.
Since 4 is greater than 2, much more is it greater than 3.
But 4 is also equal to 3, which is absurd.

The latter is the method adopted by Euclid.

Case II.—By producing the sides of one of the isosceles triangles we are enabled to make a comparison of the angles similar to that in Case I. For angle (1) ECD = angle (2) FDC, because they are the angles on the other side of the base of the isosceles triangle ACD. Proceed as in Case I.

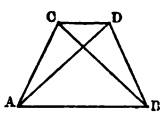


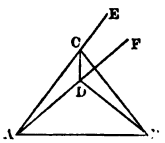
Case III. has already been considered, hence the theorem is completely demonstrated.

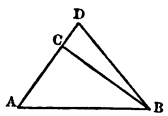
Summary of Proposition VII., Theorem 4.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated in one extremity of the base equal, and likewise those terminated at the other.

Cons.—Cases I. and II. Join the vertices of the Δs.
In Case II. produce AC and AD.







Proof (*Indirect*).—Show that $\angle$ BDC is both equal to (I. 5) and greater than BCD.

(In Case II., by means of the angles ECD and FDC on the other side of the base). Case III. is obvious.

Proposition VIII. shows that two triangles are equal if the three sides of the one are equal to those of the other each to each. This is the second case of the equality of triangles demonstrated by Euclid. The proof depends upon the last proposition, which is introduced with the express object of leading to it. The method of superposition is employed as in I. 4.

Apply one triangle to the other, and their bases being equal, will coincide. It follows that the other sides must coincide; for, if not, the condition exists which has just been shown to be impossible (I. 7).

Therefore the triangles coincide and are equal in all respects (I. 4).

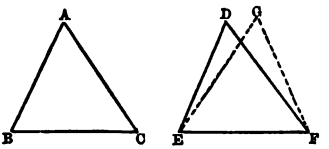
Euclid only calls attention to the fact that their vertical angles are equal, because he wishes to draw attention to that particular circumstance in order to prove the next proposition.

SUMMARY OF PROPOSITION VIII., THEOREM 5.

If in two triangles the three sides of one are equal to the three sides of the other each to each, the triangles are equal in all respects.

Cons.—Nil.

Proof.—By superposition. If the triangles do not coincide I. 7 is contradicted.



The first group of theorems is now concluded; by their study we have added some important geometrical truths to those laid down in the axioms. These may be thus summed up:—

(1) In triangles equal angles are opposite to equal sides.

Hence an isosceles triangle has two equal angles, and, conversely, a triangle with two equal angles is isosceles; an equilateral triangle is equiangular, and an equiangular triangle is equilateral (I. 5 and 6 cor.).

(2) Triangles are equal in all respects, if in the triangles:

(α) Two sides and the angle between them are equal each to each.

(β) The three sides are equal each to each.

LECTURE VI.

GEOMETRICAL ANALYSIS AND SYNTHESIS.—PROPS. 9–12.—TABLE OF RESULTS.—LOCI.

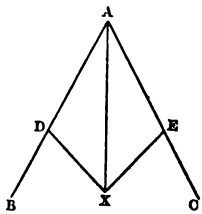
In the previous lessons we have not only been following Euclid's teaching, but it has been our aim to discover for ourselves the demonstrations of the theorems and problems proposed by him. In working out these we endeavoured to connect what we were required to prove with some other truths already established. One rule is often of great value in the investigation of a proposition, viz. to assume what we wish to prove, that is that the theorem is true or the problem solved, and to draw a diagram accordingly. The figure thus obtained will frequently suggest the lines by which the necessary construction may be effected. This method of proceeding is called analysis or resolution, and is pursued by geometers when they are required to find any new demonstration.

After having worked out the proposition we retraced our steps, arranging and dividing it into its proper parts. This proceeding is termed synthesis, a Greek word, meaning a 'putting together,' an arrangement. Euclid's propositions are all examples of this method. Thus it will be seen that synthesis is the clearest method of stating a demonstration for the instruction of pupils or the conviction of opponents, while analysis is the means by which the demonstration is discovered.

We will attempt the analysis of Prop. IX., availing ourselves of the rule just given.

It is required to bisect a given angle—that is, to divide

it into two equal angles. Draw any angle BAC and a line AX which, we will suppose, bisects it. Then angle BAX = angle CAX. To observe any results from the equality of these angles they must be included in triangles. Take AD and AE, equal parts of the arms of the angle (I. 3), and join D and E to the point X. The triangles thus formed are equal (I. 4), for DA = AE (cons.).



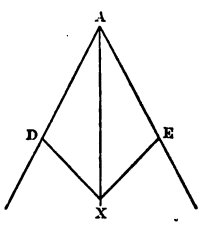
AX is common to both,

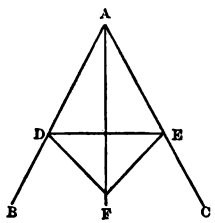
and the angles included by these lines are equal by hypothesis. Hence, if any angle is bisected, by joining equal arms of the angle with the bisecting line, equal triangles are formed.

Can we reverse this order and construct equal triangles whose vertical angles shall together make up the angle BAC? To do this we must fulfil the conditions either of I. 4 or of I. 8. As we have no angles given equal, we must follow I. 8 and endeavour to construct two triangles having each of their three sides equal.

As before, by I. 3 we can make DA = EA.

How can we obtain lines corresponding to DX and EX, *i.e.* two equal lines drawn from fixed points (D and E) and meeting at a point to be found? Since something is to be done, the postulates or the three problems we have learnt must help us. Examining the latter, we find that in Prop. I., 'to describe an equilateral triangle,' we were given two fixed





points, the extremities of the line AB, and from these we drew two equal lines meeting in a point C, not given. This

exactly meets the present case, but before we can apply Prop. I. we require a base on which to describe the equilateral triangle.

Glancing at the figure you will see that this can be obtained by joining D and E, the points from which the equal lines are to be drawn.

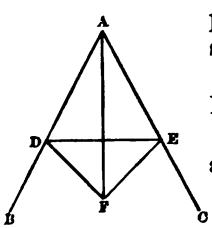
Draw this line, and on it describe the equilateral triangle DFE (I. 1). The required triangles will be completed by joining FA, for in them:—

DA = EA, cons.
DF = FE, def. 24.
AF is common.

Therefore their vertical angles DAF, FAE, are equal, and the given angle is bisected.

Summary of Proposition IX., Problem 4.

To bisect a given rectilineal angle.



Cons.—Cut off from the arms of the given angle equal parts, AD, AE. Join DE. On it describe the equilat. △ DFE. Join AF.

Proof.—The sides of the △s DAF, EAF, are equal (I. 8).

Therefore their vertical ∠s are equal, and ∠ BAC is bisected.

[Note.—In the construction of the equilateral triangle, the words 'on the side remote from A' have been inserted into the text of Euclid, because if the triangle were described on the same side as A, the points F and A might coincide, in which case the construction would fail; it holds good, however, if F falls either beyond or within A.]

Prop. X.—The next problem is to 'bisect a given finite straight line.' Turning to the figure of the last proposition, you will observe that the base of the equilateral triangle is

cut by the line AF at a point which we will call X, and that the equilateral triangle is divided by it into two equal triangles, DFX, EFX, for DF = EF (def. 24).

FX is common.

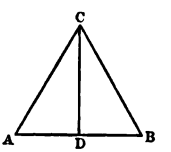
Angle DFX = angle EFX (I. 4).

(Since they are corresponding angles of triangles DFA, EFA, shown to be equal.)

Therefore the bases DX and EX are equal, and the line DE is bisected; and also since the angle DFX = EFX, the angle DFE is bisected.

Again, the same line which bisects angle DFE bisects the line DE.

These considerations will enable us to solve the problem thus: on the given line describe an equilateral triangle (I. 1). Bisect its vertical angle (I. 9), producing the bisecting line to cut the base, *i.e.* the given line, which will then be itself bisected, as we have seen.



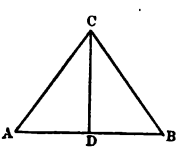
SUMMARY OF PROPOSITION X., PROBLEM 5.

To bisect a given finite straight line.

Cons.—On given line describe an equilat. Δ (I. 1). Bisect its vertical angle (I. 9), making the bisecting line cut the base.

Proof.—The two Δs into which ABC is divided are equal (I. 4);

∴ their bases are equal, and the given line is bisected.



PROP. XI.—In the next problem it is required to draw a straight line at right angles to a given straight line from a given point in the same.

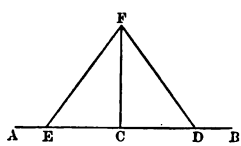
From definition 10 we learn that 'when a straight line standing on another makes the adjacent angles equal, each of these angles is called a right angle.'

Look at the diagram in the last proposition. Since the

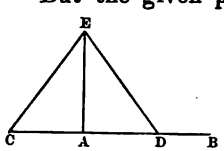
triangles ADC and BDC were shown to be equal, the angles ADC and BDC are equal; and therefore, according to the definition just quoted, they must be right angles, and DC is drawn perpendicular to AB. Then supposing AB to be the given line, and D the given point, DC is the perpendicular required. But AB in the last proposition was bisected at D, and the given point is not necessarily the point of bisection. Produce AB in one or both directions, and you will see that the construction still holds good. It remains for us to apply it to the problem we are engaged upon.

Let AB be the given line, and C the given point.

Since the given point bisects the base of the equilateral triangle we must mark off equal parts, CD, CE, on each side of the point C (I. 3). DE will then form the required base bisected at C, on it describe the equilateral triangle DFE. A line joining the vertex of this triangle with the given point will be the perpendicular required.



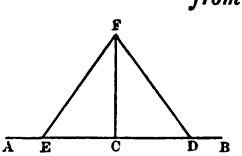
But the given point may be at one extremity of AB, as at A, in this case we must produce the line any distance beyond A, marking off on the other side of it a part equal to the line produced.



One case alone is noticed by Euclid.

SUMMARY OF PROPOSITION XI., PROBLEM 6.

To draw a straight line at right angles to a given straight line from a given point within it.



Cons.—Cut off equal parts DC, EC, on each side of the given point (I. 3). On DE (DC + EC) describe an equilat. △ DFE (I. 1). Join FC.

Proof.—△ DFC = △ EFC (I. 8)

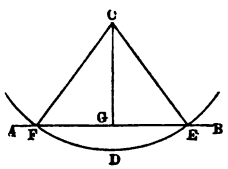
∴ ∠ DCF = ∠ ECF (and they are adjacent ∠s)

∴ FC is perpendicular to AB (def. 10).

Proposition 12 is closely related to the last; it requires us to draw a line at right angles to another from a given point without it. Euclid further specifies that the given line shall be of unlimited length; if not, the point might be so placed that it would be impossible to let fall a perpendicular from it to the given line, $\overline{a \quad b}$ ·^{c}. In the figure of I. 11, supposing F to be the given point, FC is the perpendicular required.

The perpendicular was obtained by joining the vertex of an equilateral triangle with the point bisecting its base. This triangle was described in order to find two equal lines drawn from fixed points within the given line to a point without it not found. The condition is now reversed, the equal lines must be drawn from the fixed point without the line, therefore we cannot employ Prop. I.

But equal lines can be obtained by means of the circle, the given point C from which the lines are to be drawn will form the centre, what will be the distance? The extremities of the equal lines must lie in AB, hence the circle must cut AB in two points. AB is of unlimited length, therefore, with C as centre and any point D on the other side of AB as distance, a circle can be drawn cutting AB (produced if necessary) in two points E and F. To these points draw the radii CE and CF. The remainder of the construction is obvious, bisect FE in G, join GC, which is the perpendicular required.

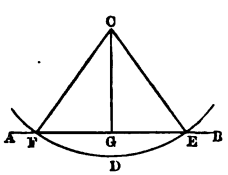


Summary of Proposition XII., Problem 7.

To draw a perpendicular to a given line from a given point without it.

Cons.—With given point as centre and any point on the other side of the line as radius, describe a circle.

Bisect that part of the line which the circle intercepts, join given point to point of bisection and to the two points at which the circle cuts AB.



Proof.—The two similar Δs are equal (I. 8);

$\therefore$ the adjacent $\angle$s FGC, EGC are equal

$\therefore$ GC is perpendicular to AB.

This proposition concludes the second group of problems, in which we have learnt:—

To bisect (α) an angle,

(β) a finite straight line.

To draw a perpendicular to a line from a point (α) within it, (β) without it.

The solution of the last proposition chiefly depended on finding two points equidistant from the given point C, and lying in AB. We observed that the circumference of a circle drawn round C would furnish an infinite number of points, satisfying the first condition, and any points in AB of course satisfy the second. The points of intersection of AB and the circle were the only ones which would satisfy both conditions.

This is called the method of loci.

In plane geometry a locus is a line, curved or straight, every point of which satisfies a certain condition.

The intersection of loci is often a ready means of determining the position of a point necessary for the solution of a problem, since straight lines will only intersect at one point and circles either with each other, or with straight lines at two.

I. 1 is an example of the intersection of loci, for a circle described round A at the distance AB is the locus of all points at that distance from A. Similarly, a circle of which B is the centre is the locus of all points at the required distance from B. One of the points at which these circles cut, *i.e.* the intersection of the loci, must form the vertex of the required triangle.

The method of loci is ascribed to the Platonic school.

LECTURE VII.

PROPOSITIONS 13–15.

A GROUP of three theorems follows, the demonstration of all of which chiefly depends on the fact that right angles are equal.

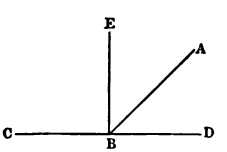
Prop. XIII. teaches us that 'the angles which one straight line makes with another on the same side of it either are, or are together equal to two right angles.

This theorem is almost self-evident, for we know by def. 10 that if the angles are equal they are right angles, and that in this case one line is perpendicular to the other.

But instead of being perpendicular let us suppose that one line is inclined a little to the other, is it not evident that while one of the angles is made larger than a right angle, the other is made proportionately smaller, and that the two angles taken together are still equal to two right angles?

It remains to show this formally. In the lesson on the definitions we noticed that Euclid did not measure his angles exactly, but took the right angle as a standard of comparison, speaking of other angles as being greater or less than a right angle. Let us apply this test here.

Let AB be a line falling upon CD so that the angles are unequal in size. To compare these angles with right angles we must draw a line perpendicular to CD from point B (I. 11).



The figure is now divided into two given angles ABC, ABD, two right angles EBC, EBD, three angles EBC, EBA, ABD. The angles EBA, ABD together make up and are therefore equal to angle EBD, to complete the figure we have angle EBC.

Therefore $\angle$ EBC + $\angle$ EBD = $\angle$ EBC + $\angle$ ABE + $\angle$ ABD, *i.e.* the two right angles = the three angles.

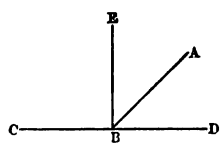
Again, angle ABC is made up of and is therefore equal to angles EBC and ABE, to complete the figure we must add $\angle$ ABD.

Then $\angle$ ABC + $\angle$ ABD = $\angle$ EBC + $\angle$ ABE + $\angle$ ABD, *i.e.* the two given angles = the three angles.

Therefore the two given angles = the two right angles (ax. 2).

Summary of Proposition XIII., Theorem 6.

The angles which one straight line makes with another straight line on one side of it, either are two right angles or are together equal to two right angles.



Case I.—When the angles are equal (def. 10).

Case II.—When the angles are unequal.

Cons.—Draw a perpendicular to CD (I. 11).

Proof.—The two given $\angle$s / The two right $\angle$s } = { the three $\angle$s into which the figure is divided.

$\therefore$ the two given $\angle$s = two right $\angle$s (ax. 2).

or $\angle$ EBC + $\angle$ EBD / $\angle$ ABC + $\angle$ ABD } = $\angle$ EBC + $\angle$ ABE + $\angle$ ABD.

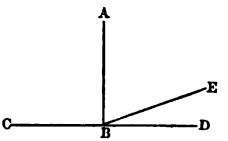
$\therefore$ $\angle$ EBC + $\angle$ EBD = $\angle$ ABC + $\angle$ ABD.—Q.E.D.

Prop. XIV.—The enunciation of the next theorem is 'If at a point in a straight line two other straight lines on the opposite sides of it make the adjacent angles together equal to two right angles, these lines shall be in one and the same straight line.'

The meaning of this will become apparent by an example. Let AB be any straight line, and let C and D be points one on each side of the point B, then supposing straight lines to be

drawn from C and D meeting at B, the theorem teaches us that if the angles ABC, ABD are equal to two right angles, CB and BD are in the same straight line.

The demonstration is a very simple example of the indirect method. Angles ABC, ABD being given equal to two right angles, we assert that CB and BD form one straight line. If BD is not in the same straight line as CB, produce CB to E (in the same straight line). Then we have the angles ABC, ABE equal to two right angles according to the last proposition. Also angles ABC, ABD given equal to two right angles. From these pairs subtract the common angle ABC, and we have remaining angle ABE equal to angle ABD, the part to the whole, which is absurd. Hence it is evident that no other line but BD can be drawn in a straight line with CB.

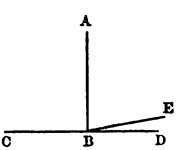


Summary of Proposition XIV., Theorem 7.

If two straight lines meet a third at a point, making the adjacent angles equal to two right angles, these two lines shall be in the same straight line.

Cons.—If BD is not in the same straight line with CB, produce CB to E.

Proof.—Both pairs of angles = 2 rt. ∠ s, α (hyp.) β (I. 13); subtract common angle, then ∠ ABD = ∠ ABE, which is absurd, &c., &c.



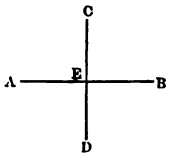
Prop. XV.—We pass to the third theorem of the group.

If two straight lines cut one another, the vertical or opposite angles are equal. The vertical angles are those whose vertices or heads are exactly opposite. In some of the old editions of Euclid they are called the head angles.

Let AB and CD be two straight lines cutting at the point E, then the vertical angles are equal, that is we have to show that:

$$\angle \text{CEB} = \angle \text{AED}$$
$$\angle \text{CEA} = \angle \text{BED}$$

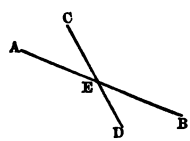
We have already said that the proof of this theorem mainly depends on the fact that all right angles are equal. Let us endeavour to apply this principle.



First let us suppose that AB and CD cut at right angles, then it is obvious that there will be four right angles in the figure and the vertical angles will be equal according to the postulate referred to (post. 4).

Next, let the two straight lines cut at any other angle, then we shall not find any single right angles, but there will be pairs of angles, which by I. 13 will be together equal to two right angles. Let us examine the figure and see how many of these pairs we can obtain.

CE	falling on	AB	makes angles	CEA, CEB=	two right angles
AE	„	CD	„	CEA, AED=	
DE	„	AB	„	AED, BED=	
BE	„	CD	„	CEB, BED=	



Since these pairs of angles are each equal to two right angles, any pair must be equal to any of the other pairs. In the first two pairs of angles in our table we have a common angle CEA; take this away, and the remaining angle CEB = the remaining angle AED, and these are vertical angles. Again, in the second and third pairs we have a common angle AED; subtract this, and angle CEA = angle BED, the other pair of vertical angles.

You will notice that the vertical angles might also be proved equal, by subtracting the common angle from the third and fourth pairs, and also from the first and fourth.

Also, since each pair of angles is equal to two right angles the four angles made by one line cutting another are together equal to four right angles. Again, suppose that besides AB and CD in the figure we have any other lines, FG, HK, meeting or cutting at the point E, though each individual angle will be smaller you will see that the whole space is the same, and that all these angles must be together equal to four

right angles. From these considerations we get two important corollaries.

I. If two straight lines cut one another, the angles which they make at the point of section are together equal to four right angles.

II. All the angles made by any number of straight lines meeting at one point are together equal to four right angles.

From this group of theorems we learn the following facts about angles :—

If one straight line meets another the resulting angles are equal to two right angles.

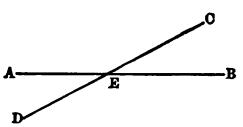
If one straight line cuts another, the angles formed are equal to four right angles, and also the vertical opposite angles are equal. If any number of straight lines meet in a point, the angles formed are equal to four right angles.

Summary of Proposition XV., Theorem 8.

If two straight lines cut one another, the vertical angles are equal.

Cons.—Nil.

Proof.—



AE falling on CD makes CEA, AED = two
DE „ AB „ DEB, AED = right angles.

From these equals subtract common angle, and the remaining vertical angles are equal. In the same way the other pair may be shown equal.

LECTURE VIII.

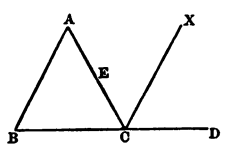
PROPOSITIONS 16–19.

PROPOSITION XVI. proves that if one side of a triangle be produced, the angle thus formed outside the triangle is greater than either of the angles which are opposite to it inside the triangle.

Let ABC be a triangle of which the side BC is produced to D: then the exterior angle ACD shall be greater than either of the interior and opposite angles CBA, BAC.

The construction made use of by Euclid for the proof of this theorem is exceedingly ingenious. We may imagine that it was arrived at in this way:—

According to the rule given in Lecture VI. for analysis, let us assume what we have to show, viz. that angle ACD is greater than angle BAC. Then we will make an angle ACX a part of ACD, which we will assume to be equal to angle BAC. To observe any results from the equality of these angles we must, as in several other propositions, include them in equal triangles. How shall we do this? We have $\angle$ BAC = $\angle$ ACX. As the length of the arm of an angle does not affect the angle, we can make CX = BA; by bisecting AC in E (as in Proposition 12) we make the other arms of the angles equal; joining XE, EB, the extremities of the equal arms, we obtain the bases. The triangles thus constructed satisfy the conditions of I. 4, and are therefore equal.

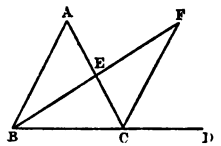


Next, can we retrace our steps, and construct these equal triangles, without assuming angle ECX?

We can, as before, bisect AC in E, thus obtaining AE equal to EC.

We can join EB, and produce it to F, making EF equal to EB.

Do we know anything about the angle contained by these equal sides? Looking at the figure you will see that angle BEA is vertically opposite to angle FEC, and therefore these angles are equal (I. 15).



Then, joining FC, we have constructed the triangles BAE, FEC equal in all respects (I. 4). Therefore the angle BAE (*i.e.* BAC) = angle ECF, a part of angle ACD. It is evident, then, that the angle ACD is greater than the angle BAC. It remains to show that angle ACD is greater than angle ABC. Produce AC to G. The angle BCG is evidently opposite the angle ACD, and is therefore equal to it (I. 15).

Then, having bisected BC with the same construction as before, we can show that angle BCG is greater than angle ABC. Therefore, the angle ACD is also greater than angle ABC. Hence we have shown that the exterior angle ACD is greater than either of the interior and opposite angles.

Summary of Proposition XVI., Theorem 9.

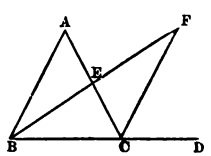
The exterior angle of any triangle is greater than either of the interior and opposite angles.

I. To show that $\angle$ ACD is greater than $\angle$ BAC.

Cons.—Bisect AC in E.

Join BE, produce to F, making EF = EB.

Join FC.



Proof. —$\triangle$ BAE = $\triangle$ ECF (I. 4).

($\because$ AEB, FEC are vertical $\angle$ s, I. 15.)

$\therefore$ $\angle$ ECF = $\angle$ BAC.

$\therefore$ $\angle$ ACD is greater than $\angle$ BAC.

II. To show that $\angle$ ACD is greater than $\angle$ ABC.

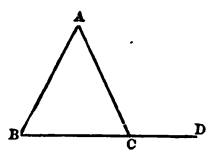
Cons.—Produce AC to G, bisect BC, &c.

Proof.—∠ BCG can be shown, in the same manner as above, to be greater than ∠ ABC.

∠ ACD = ∠ BCG (I. 15), and ∴ is greater than ∠ ABC.

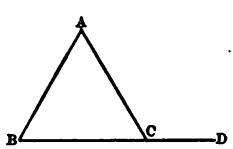
Prop. 17. A simple theorem follows this somewhat intricate demonstration: 'Any two angles of a triangle are together less than two right angles.'

To show this we must compare them with two angles which either are or are together equal to two right angles. These are easily obtained, as you will see from a glance at the last figure, by producing one side of the triangle; then the exterior angle, together with the adjacent interior angle, are equal to two right angles (I. 13). Produce BC, the base of triangle ABC, to D, then angles ACD, ACB are together equal to two right angles. Now we know that angle ACD is greater than either of the interior opposite angles; it follows, then, that angle ACB and either of these (interior and opposite) angles must be together less than two right angles. But are these two opposite angles, taken together, less than two right angles? If we produce another side, *e.g.* AB to E, we shall find that the same proof applies, and that they are also less than two right angles.



SUMMARY OF PROPOSITION XVII., THEOREM 10.

Any two angles of a triangle are together less than two right angles.



Cons.—Produce one side of △.

Proof.—The ext. ∠ and adjacent int. ∠ are greater than adjacent int. ∠ and either of opposite int. ∠s (I. 16), and = two rt. ∠s (I. 13) ∴ either of these pairs of int. ∠s are less than two rt. ∠s.

Produce another side.

Repeat proof for the remaining pair of angles.

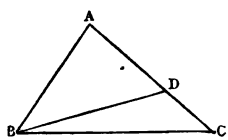
We will conclude this lesson with two short theorems, pointing out another relation between the sides and angles of a triangle. These are the converse of each other.

(α) The greater side of every triangle has the greater angle opposite to it (I. 18).

(β) The greater angle of every triangle has the greater side opposite to it.

Let ABC be any triangle whose side AC is greater than AB. We have to show that angle ABC is greater than angle ACB. What do we already know about the relation of sides to angles in the same triangle? From the 5th and 6th propositions we learn that equal angles are opposite equal sides. Let us compare an isosceles triangle with the one given.

Since AC is greater than AB, cut off AD = AB (I. 3). Join BD, then ABD is an isosceles triangle. Can we observe any relation between the equal angles of the isosceles triangle and those which we are required to compare? First, each of these equal angles is less than angle ABC, for one of them is a part of ABC (ax. 9).

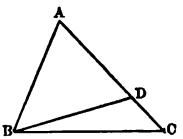


Has either of them any relation to angle ACB? Yes; angle ADB is the exterior angle of the triangle DBC, and is therefore greater than the interior and opposite angle ACB.

Then angle ADB is less than angle ABC and greater than angle ACB, 'much more then must angle ABC be greater than angle ACB.'

SUMMARY OF PROPOSITION XVIII., THEOREM 11.

The greater side of every triangle has the greater angle opposite to it.



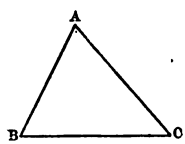
Cons.—Cut off from greater side a part equal to less. Join DB.

Proof.—$\angle$ ADB = $\angle$ ABD (I. 5), is less than $\angle$ ABC.

$\angle$ ADB is greater than int. and opposite $\angle$ ACB (I. 16).

$\therefore$ $\angle$ ABC is greater than $\angle$ ACB.

I. 19. The converse theorem, as is often the case, is proved by the indirect method. Angle ABC being given greater than angle ACB, we have to show that side AC is greater than the side AB.



If AC is not greater than AB, it must either be equal to it or less.

What would be the consequence if AB = AC ?

We know that $\angle$ ABC would be equal to $\angle$ ACB (I. 5), but it is not (being given greater).

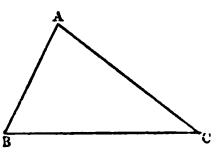
What would be the consequence if AC is less than AB ?

We know that the angle ACB would be greater than the angle ABC (I. 18), but it is not.

It remains, then, that AC must be greater than AB.

SUMMARY OF PROPOSITION XIX., THEOREM 12.

The greater angle of every triangle has the greater side opposite.



Cons.—Nil.

Proof (*Indirect*).—AB is not equal to AC (I. 5).

AB is not greater than AC (I. 18),

$\therefore$ it is less.

LECTURE IX.

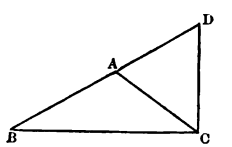
PROPOSITIONS 20–21.

I. 20. We come to the well-known and much-derided 20th Proposition, said by the Epicureans to be 'evident even to asses.' 'Any two sides of a triangle are greater than the third.' Though this fact seems very obvious, its mathematical proof is not easy, besides which, this and other plain propositions ought to be proved in order to avoid increasing the number of axioms.

Let ABC be any triangle, we will endeavour to show that sides BA, AC are together greater than BC.

In order to compare these lines readily, make one line equal to the two BA, AC, by producing BA to D, making AD equal to AC (I. 3). Then BD will represent the lines BA, AC, and we have to show that BD is greater than BC.

In the last proposition one line was shown to be greater than another because it was opposite a greater angle of a triangle. BD, BC will be sides of a triangle if we join DC.



Let us examine the angles in the figure thus obtained, and see if we can arrive at this proof.

Is angle BCD greater than angle BDC ?

$\angle$ BCD is greater than $\angle$ ACD, a part of it.

$\angle$ ACD = $\angle$ ADC (*i.e.* $\angle$ BDC) $\because$ AD = AC.

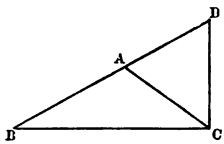
$\therefore$ $\angle$ BCD is greater than $\angle$ ADC (BDC).

$\therefore$ BD is greater than BC, and BD = BA and AC.

We can repeat this proof to show that BC, CA are greater than BA, and AB, BC greater than AC, and then the theorem is fully demonstrated.

SUMMARY OF PROPOSITION XX., THEOREM 13.

Any two sides of a triangle are together greater than the third.



Produce BA to D, make AD = AC. Join DC.

∠ ACD = ∠ ADC (I. 5).

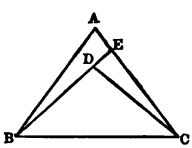
∠ BCD is greater than ∠ ACD, and consequently than ∠ ADC.

∴ BD (= BA, AC) is greater than BC (I. 19).

Repeat demonstration for the other pairs of sides.

I. 21. We pass to Prop. XXI., the last of this group of theorems. If from the ends of the side of a triangle there be drawn two straight lines to a point within it, these will be less than the other sides of the triangle, but will contain a greater angle.

Let ABC be any triangle, and from B, C, ends of side BC, draw straight lines BD, CD to a point D within the triangle, BD and DC shall be less than BA and AC, but the angle BDC shall be greater than the angle BAC.



I. To show that BD, DC are less than BA, AC. In order to connect these two pairs of lines, produce one of the lines meeting at D to meet the opposite side of the triangle BAC in E.

We have now BA + AC = BA + AE + EC (A).

BE + EC = BD + DE + EC (B).

BD + DC (C).

Let us compare these groups of lines, beginning at the bottom of the chain.

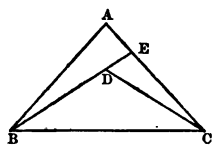
DC must be less than DE + EC, for any two sides of a triangle are greater than the third, and BD is common to groups B and C, therefore BD + DC are less than BE + EC.

Again, BE is less than BA + AE (I. 20), and EC is

common to groups A and B; therefore BE + EC are less than BA + AC; and since we have shown BD + DC to be less than BE + EC, they must also be less than BA + AC.

II. To show that the angle BDC is greater than the angle BAC.

We know that one angle may be shown to be greater than another if in any triangle it has the greater side opposite (I. 19); or, if the two angles are not in the same triangle, it can be shown to be the exterior angle of a triangle, of which the other is an interior and opposite angle (I. 16). The latter condition will be the most likely to help us here. Examining the figure you will see



angle BDC is greater than ∠ BEC (I. 16).
„ BEC „ ∠ BAC (I. 16).
∴ ∠ BDC „ ∠ BAC.

You will observe that in this theorem, as in one or two previous, we were unable to compare the lines and angles in question *directly* with each other, but a third set of lines and a third angle became, as it were, our medium of comparison. This method is often very useful in mathematical demonstrations.

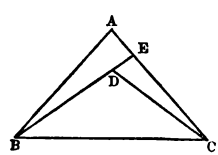
Summary of Proposition XXI., Theorem 14.

If from the ends of the base there be drawn two straight lines to a point within a triangle, these lines will be less than the other two sides of the triangle, but will contain a greater angle.

Cons.—Produce BD to meet AC in E.

Proof.—I. To show that BD, DC are less than BA, AC. DE, EC are greater than DC (I. 20), add BD.

Then BE, EC are greater than BD, DC.



Similarly BA, AC are greater than BE, EC, and consequently than BD, DC.

II. To show that angle BDC is greater than angle BAC.

Ext. $\angle$ BDC is greater than int. and opposite $\angle$ DEC (I. 16).

Ext. $\angle$ DEC is greater than int. and opposite $\angle$ BAC.

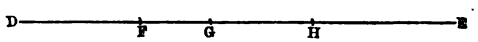
$\therefore$ $\angle$ BDC is greater than $\angle$ BAC.

The group of theorems which we have just concluded extends from Props. 13–21, and may be divided into two sections. (α) Props. 13, 14, 15, which treat of the angles made by straight lines either meeting or cutting each other. The results of these have been already summed up (v. p. 45). (β) Props. 16–21 inclusive, treating of various properties of triangles; the results obtained in these we shall discuss a little later on.

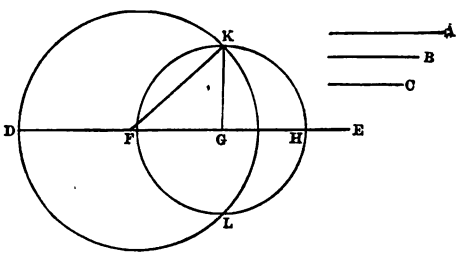
LECTURE X.

PROPOSITIONS 22–23.

I. 22. Two problems follow. Prop. 22 requires us to construct 'a triangle, of which the sides shall be equal to three given straight lines, but any two of these must be greater than the third.' Let A, B, C represent the three given lines, then, in order that we may have a connected measure for these lines, let us mark off on a line of unlimited length DF = line A, FG = B, and GH = C, then for the base of the triangle we can take any one of these portions. You will remember that in the very first problem we



had to deal with, we were required to construct a triangle (equilateral); the base was then given us, and we were able to make the other two sides by drawing circles which cut each other. Can we apply this method here? Yes, for having chosen our base, we can take the two remaining portions as



radii of two circles. But in order to make a triangle it will be necessary that these circles should cut as in the figure

LECTURE XI.

PROPOSITIONS 24–26.

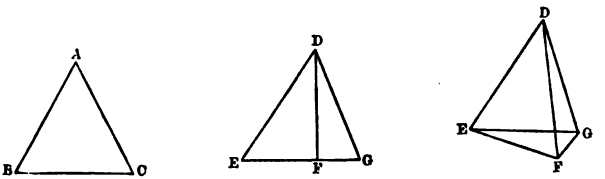
THE next two theorems teach us another relation between the sides and angles of triangles; *i.e.* the relation between the base and the vertical angle of a triangle. Supposing the two sides of a triangle to be of a certain length, the greater the vertical angle is, the greater the base will be, and conversely.

I. 24. Euclid enunciates the first of these theorems thus: If two triangles have two sides of the one equal to two sides of the other each to each, but the angle contained by the two sides of the one (*i.e.* the vertical angle) greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let us take two triangles ABC, DEF, having their sides BA, AC = to the sides DE, DF, but let the angle BAC be greater than the angle EDF, we have to show that the base BC is greater than the base EF.

The first thing that will strike you is, that in order to compare these triangles we must connect them in some way; in the 4th and 8th props., in which two triangles are compared with each other, we employed the method of superposition. Now instead of supposing one triangle to be taken up and placed on the other, a method which we have seen is sparingly used by Euclid, by the help of the last two props. we can construct a triangle equal to ABC and having an angle EDG equal to the angle BAC, and as DE and AB are equal, we can make DE one side of the new triangle, drawing DG equal to AC, and since the angle EDG = angle BAC, the base

EG will be equal to BC. So that the triangle EDG may be taken to represent the triangle ABC, and we have to show that EG is greater than EF.



In drawing the figure we have just described you will find that the point F may fall on EG, so that the base E F forms a part of EG, or it may fall without the triangle EDG so that EF is below EG, or lastly it may fall within triangle EDG so that EF is above EG. (See fig. below.) The case in which F falls on EG needs no demonstration, for EG is necessarily greater than its part EF.

Let us consider the second case in which EF falls below EG, then we may connect EF and EG in one triangle by joining FG. And we know that the greater side of any triangle is opposite the greater angle (I. 19). Is angle EFG greater than angle EGF?

Examining the angles we find that:—

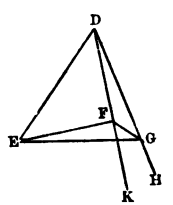
angle EFG is greater than DFG (its part)

DFG = DGF (I. 5)

DGF is greater than EGF.

Therefore angle EFG is greater than EGF, and EG (*i.e.* BC) is greater than EF.

Lastly we have the third case in which DF falls above DG, joining FG a triangle is formed of which EF and EG are sides; but in order to follow the same argument as in Case II. we must produce the equal sides DF, DG to K and H, then the angles on the other side of the base of the isosceles triangle DFG become the medium of comparison, and the rest of the proof is identical with that of Case II.



This case may be very simply proved without any construction by the help of Prop. 21; for since DF, FE are drawn 'from the ends of one side of a triangle to a point within it,' they are less than DG, GE, but DF = DG, therefore EF is less than EG.

Euclid himself only notices the second of these cases. In order to remedy this imperfection, Dr. Simson in his edition reduces the three cases to one, by inserting into the text the words, 'of the two sides DE, DF, let DG be the side which is not greater than the other.' This condition ensures EF falling below EG, for, says Dr. Simson, 'it is easy to perceive that DG being equal to DF, the point G is in the circumference of a circle described from the centre D at the distance DF, and must be in that part of it which is above the straight line EF, since the angle EDG is greater than the angle EDF.' We will now pass to the summary of this proposition, taking the three cases, which seems to be on the whole the most satisfactory method. If you wish to follow Dr. Simson, you must remember to make the condition above referred to, that DE is not greater than DF.

Summary of Proposition XXIV., Theorem 15.

If two triangles have two sides of the one equal to two sides of the other, but the angle contained by the two sides of the one greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle must be greater than the base of the other.

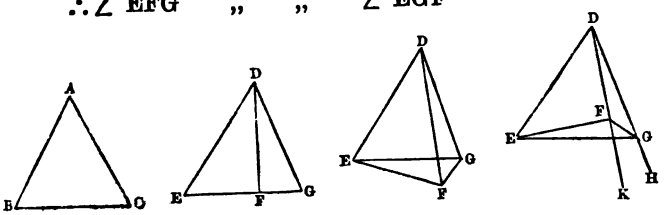
Cons.—At point D in △ DEF make ∠ EDG = ∠ BAC, making DG = AC (I. 23). Join EG, GF. Case III. produce DF and DG to H and K.

Proof.—△ DEG = △ ABC (I. 4).

Case I. When EF falls on EG;
EF is a part of EG ∴ EG is greater than EF.

Case II. When EF falls below EG;
∠ DFG = ∠ DGF (I. 5)

$\angle$ DGF is greater than $\angle$ EGF
$\therefore \angle$ DFG ,, ,, $\angle$ EGF
$\therefore \angle$ EFG ,, ,, $\angle$ EGF



$\therefore$ EG is greater than EF (I. 19), but EG = BC
$\therefore$ BC is greater than EF.

Case III. $\angle$ GFK = $\angle$ FGH, and proceed as in Case II.

I. 25. The converse proposition follows, which is demonstrated, as is usually the case, indirectly. The triangle with the larger base has the larger vertical angle, if not either I. 4 or the last proposition is contradicted.

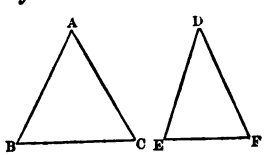
Summary of Proposition XXV., Theorem 16.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

Cons.—Nil.

Proof (Indirect).—

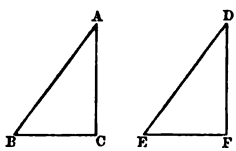
$\angle$ BAC is not equal to $\angle$ EDF (I. 4),
,, is not less than ,, (I. 24),
$\therefore$ it is greater.—Q.E.D.



Proposition XXVI. demonstrates the third case of the equality of triangles considered by Euclid. It shows that

triangles are equal, if in each triangle two angles and one side are equal to the corresponding angles and side in the others. The proof, as in the other propositions treating of the equality of triangles, is indirect.

Let us take two triangles, ABC, DEF, of which the angles ABC, BCA, are severally equal to the angles DEF, EFD. You will observe that the equal sides may be related to the equal angles in two ways: they may lie between them as the sides BC and EF, or they may be opposite to one of the equal angles, as either of the other pairs of sides.



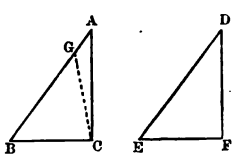
These relations of sides and angles give two cases in the theorem requiring a separate proof.

I. Let us take the case in which the equal sides lie between (or 'are adjacent' to) the equal angles.

We have, as you will remember, two means of showing the equality of triangles, *i.e.* by satisfying the conditions of I. 4, or of I. 8.

Since the angles at B and C are equal to those at E and F, and BC = EF, if we can show that either of the other pairs of sides are equal, we shall have the conditions required by I. 4, and it follows that the triangles in question are equal.

Let us then endeavour to show that AB = DE, or, as we are to follow the indirect method, that some absurdity will result if these lines are not equal. If AB is not equal to DE, one of them must be greater than the other; let AB be the greater, and from it cut off a part GB equal to DE. Then by joining GC it is plain that a triangle GBC is formed equal in all respects to DEF (I. 4), for we have GB, BC = DE, EF, and the included angles were given equal.

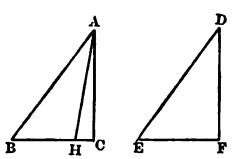


Since the triangles GBC and DEF are equal in all respects, the angle GCB = angle DFE, but we know also that

angle DFE = angle ACB (hyp.). Therefore angle GCB = angle ACB, the part to the whole, which is impossible. Hence it is evident that AB and DE must be equal, since an absurdity follows immediately from the supposition that they are unequal, and this being the case we have the other pair of sides required for proving the two triangles ABC, DEF, equal by (I. 4).

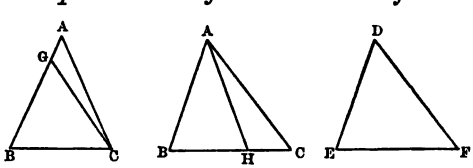
AB, BC = DE, EF
angle ABC = angle DEF
∴ triangle ABC = triangle DEF.

We must now consider the second case, in which the equal sides are opposite equal angles. The same angles being equal, let us take side AB = DE. Then in order to meet the conditions of I. 4 we shall have to show that BC = EF. Adopting a similar construction to that of Case I., let us assume that BC is the greater, cutting off BH = EF and joining HA. Then, as in Case I., by I. 4 the triangles BHA, DEF, are equal, and the angle BHA being equal to EFD = ACB; but the impossibility resulting differs, for BHA is not a part of ACB, but you will observe that it is the exterior angle of the triangle HCA, and therefore cannot be equal to the interior and opposite angle at C. Therefore BC = EF, and the proof concludes as in the first case.



Summary of Proposition XXVI., Theorem 17.

To show that triangles are equal if two angles and one side of the one are equal to two angles and one side of the other.



Case I. In which the equal sides lie between the equal angles.

Case II. In which the equal sides are opposite the equal angles.

Cons.—Cut off from side assumed to be greater a part equal to the corresponding side in the other triangle, join the extremity of this line with opposite angle.

Proof (*Indirect*).—The new triangle thus formed in each case = $\triangle$ DEF (I. 4), one of its angles (Case I. $\angle$ GCB, Case II. $\angle$ BHA) = $\angle$ BCA, which involves the absurdity (Case I.), that the part = whole (Case II.), that ext. $\angle$ of $\triangle$ = interior and opposite.

Hence AB = DE (Case I.), and BC = EF (Case II.), and triangles ABC, DEF are equal (I. 4).—Q.E.D.

This proposition is also ascribed to Thales.

It has been proposed to prove the first case by the method of superposition. Let $\triangle$ ABC be applied to $\triangle$ DEF so that BC falls upon EF.

Then since BC and EF are equal they will coincide; and since the angles at the bases of the two $\triangle$s are equal, each to each, the sides AB, AC will fall upon DE, DF (produced if necessary).

$\therefore$ The point A must coincide with point D, the only point common to DE and DF.

Therefore the $\triangle$s coincide and are equal in all respects.

LECTURE XII.

ANALYSIS OF THE FIRST SECTION OF BOOK I. — INTRODUCTION TO THE THEORY OF PARALLEL STRAIGHT LINES.—PROPS. 27–29.

We have reached the conclusion of the first division of Book I. of the Elements, treating chiefly of the relations and properties of triangles. We will classify and enumerate the results obtained in these twenty-six propositions.

I. Of the relations of triangles to each other.

We have learnt that triangles are equal in all respects, when the following parts in two or more triangles are equal:—

(α) Two sides and the included angle (I. 4).

(β) Three sides (I. 8).

(γ) Two angles and the side between them, or the side opposite one of them (I. 26).

II. Of the properties of triangles.

(α) The relations of sides to angles.

Equal angles are opposite to equal sides, and conversely (I. 5 and 6).

Hence an isosceles triangle has two equal angles, and an equilateral triangle has three, and if a triangle has two equal angles we know that it must be isosceles, if three, equilateral.

The greater side of any triangle is opposite to the greater angle, and conversely (I. 18 and 19).

The other sides remaining the same, the greater the vertical angle the greater the base, and conversely (I. 24 and 25).

(β) The relations of angles to angles.

The exterior angle of any triangle is greater than either of the interior opposite angles (I. 16).

Any two angles of a triangle are together less than two right angles (I. 17).

(γ) The relations of sides to sides.

Any two sides of a triangle are together greater than the third (I. 20).

On the same base, and on the same side of it, there cannot be two triangles having their sides terminated in one end of the base equal, and *likewise* those terminated in the other (I. 7).

If two triangles be drawn on the same base, the vertex of one of which falls within the other, the two sides of the triangle whose vertex falls within the other are together less than those of the other triangle, but contain a greater angle (I. 21).

(δ) Two problems relating to triangles.

To construct an equilateral triangle on a given line (I. 1).

To construct a triangle having its sides equal to three given lines, of which, in accordance with I. 20, any two must be greater than the third (I. 22).

In all sixteen propositions; of the remaining ten, five propositions relate to angles.

We know how—

To draw a rectilineal angle equal to a given angle (I. 23).

To bisect a given angle (I. 9).

We know—

That the angles made by one straight line falling on another = two right angles (I. 13, and its converse I. 14).

That the angles made by two straight lines cutting one another, or by any number of lines meeting in a point = four right angles.

Also that when two straight lines cut one another the vertical opposite angles are equal (I. 15, and cor.).

Five propositions relate to straight lines.

We know how—

To draw a straight line from any point equal to a given straight line (I. 2).

To cut off from the greater of two straight lines a part equal to the less (I. 3).

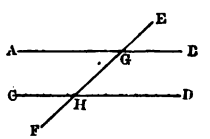
To bisect a given finite straight line (I. 10).

To draw a perpendicular to a straight line from a given point within (I. 11) or without it (I. 12).

The second section of Book I. deals with a very important and rather difficult portion of geometry—the theory of parallel straight lines. In it def. 35 and post. 6 are both used for the first time, and as they form the groundwork of Euclid's method of dealing with this subject, it will be well to consider them attentively.

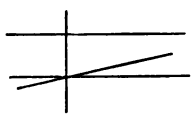
The definition referred to tells us that 'parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet'—that is, they always remain at exactly the same distance apart.

In the various theorems which treat of parallel lines we shall be very much concerned with the angles which are made when one straight line cuts two parallels. Let us take any straight line EF, cutting parallel lines AB CD in G and H, you will see that it makes with them eight angles, which, for convenience in reference, are given different names, according to their situation. The four angles which are contained between the parallels, two on each side of the cutting line, are called the interior angles. Those outside of the parallels, two above and two below, are called *exterior angles*. Again, the interior angles, taken in pairs thus, AGH GHD, and BGH GHC, are called alternate angles.

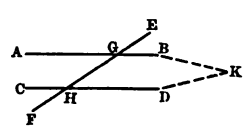


The four '*interior* angles' are equal to four right angles (I. 13). It is with these that the sixth postulate deals. It tells us that 'If a straight line meets two other straight lines so as to make the two interior angles on the same side of it, taken together less than two right angles, these lines, being continually produced, shall at length meet upon that side on

which are the angles which are together less than two right angles.' This postulate has been objected to as not being self-evident, and various attempts have been made by mathematicians to adopt some simpler principle, but without much success; and on a careful examination of the postulate in question you will see that the chief difficulty lies in the length at which it is expressed. If we draw two parallel lines and a line cutting them at right angles, then the two interior angles taken on one side of the line will evidently be right angles. Now, if we draw a third line, cut by the perpendicular and making one of the interior angles less than a right angle, it will be quite evident that this line will converge towards one of the parallels in that direction in which 'the two interior angles are together less than two right angles' (in the figure one right angle + one less angle). It has been proposed by Dr. Playfair and other mathematicians to supersede this postulate by one derived from the figure which we have just drawn, viz.: Two straight lines which cut one another cannot both be parallel to the same straight line. This is, at least, easier to remember, and necessitates but a very slight alteration in Euclid's text, which will be pointed out when we reach Proposition XXIX., in which it is used.



I. 27. After this preface we can discuss the first theorem of the division, which tells us that 'when a straight line falling on two other straight lines makes the alternate angles equal, these two lines shall be parallel to each other.' This is proved by the indirect method. Let AB and CD be two straight lines cut by the line EF at G and H so that the alternate angles AGH GHD are equal; we have to show that AB is parallel to CD.

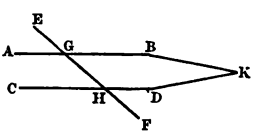


Let us ask ourselves, what would be the result supposing that these lines are not parallel? Parallel lines, we know, will never meet, but lines which are not parallel, being produced far

enough, will meet. Then, if AB and CD are not parallel, they will meet if produced far enough. Let them meet towards B and D at the point K. Then, as these straight lines meet at a point they form an angle GKH, and as the arms of this angle are cut by EF at GH a triangle GHK is formed. Do we know anything about the sides or angles of this triangle? We know that one of its angles GHK (or GHD) was given equal to angle AGH. Also we can see that AGH is the exterior angle of the triangle GHK, and we know that it must therefore be greater than either of the interior and opposite angles of which GHK is one. But it is not, for it is equal to GHD; therefore, GHK is not a triangle, and it is evident that lines AB and CD do not meet when produced towards B and D. Similarly we can show that they cannot meet towards A and C. Therefore they are parallel, according to def. 35.

Summary of Proposition XXVII., Theorem 18.

If a straight line falling on two others make the alternate angles equal, these lines shall be parallels.



Cons.—Let AB and CD meet towards B and D, forming a Δ GHK.

Proof (*Indirect*).—

$\angle$ AGH = $\angle$ GHD (or GHK) (Hyp.)

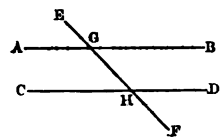
$\therefore$ ext. $\angle$ AGH = int. and opposite $\angle$ GHK of Δ GHK, which is impossible.

Repeat proof for the other side.

Therefore, AB and CD will never meet, and are parallel. —Q.E.D.

I. 28. The next proposition contains two short theorems. If two straight lines are cut by a third which makes (α) the exterior angle equal to the interior and opposite on the same side of the line, or (β) the two interior angles on the same side, together equal to two right angles, these lines are parallel.

Taking lines AB and CD, cut by EF as in the last proposition, the exterior angle EGB being equal to the interior and opposite angle GHD, we have to show that AB is parallel to CD. We are now able to show that lines are parallel in two ways, by the indirect method adopted in the last proposition, and also directly, if the alternate angles are equal. Let us try to establish this fact. Angle EGB is given equal to angle GHD, one of the alternate angles; is there any reason why it should be equal to the other? Yes; for EGB and AGH are opposite vertical angles, and are therefore equal by I. 15. Then, since angle EGB is equal to each of the alternate angles, these angles must be equal to one another (ax. 1), and AB is therefore parallel to CD. In the second part of the theorem we have to show that since angles BGH, GHD are together equal to two right angles, AB is parallel to CD. As in the first case, let us endeavour to show that the alternate angles are equal. We are given angles BGH + GHD = two right angles, that is, angle BGH + one of the alternate angles. Hence it is clear that if angle BGH and the other alternate angle are equal to two right angles, the alternate angles must be equal. Is there any reason why angles AGH and BGH should be together equal to two right angles? Yes; by Proposition XIII. we know that the straight line GH, falling upon AB, makes the adjacent angles equal to two right angles.



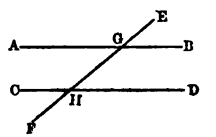
We have, then, these two equations:—

$$\left.\begin{array}{l} \angle AGH + \angle BGH \text{ (I. 13)} = \\ \angle GHD + \angle BGH \text{ (Hyp.)} = \end{array}\right\} \text{two right angles.}$$

Taking away the common angle we have remaining $\angle AGH = \angle GHD$, and these are alternate angles. Therefore AB is parallel to CD.

SUMMARY OF PROPOSITION XXVIII., THEOREM 19.

If a straight line falling on two others make the exterior angle equal to the interior and opposite on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.



Pt. I., Cons.—Nil.

Proof.— $\angle$ EGB = $\angle$ AGH (vertical opposite angles (I. 15).

$\angle$ EGB = $\angle$ GHD (Hyp.)

$\therefore$ $\angle$ AGH = $\angle$ GHD (ax. 2), and they are alternate angles.

$\therefore$ AB is $\parallel$ to CD.

Pt. II.— $\angle$ BGH + $\angle$ GHD = 2 rt. $\angle$s (Hyp.)

$\angle$ BGH + AGH = 2 rt. $\angle$s (I. 13)

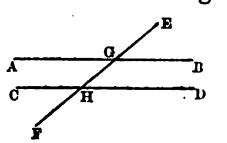
$\therefore$ $\angle$ GHD = $\angle$ AGH (alternate angles).

$\therefore$ AB is $\parallel$ to CD.

Proposition XXIX., the converse of the last two, shows that a straight line falling on two parallels makes (α) the alternate angles equal, (β) the exterior angle equal to the interior and opposite on the same side of the line; the two interior angles on the same side together equal to two right angles.

The proof of the first part of this theorem is indirect and involves the use of the sixth postulate already referred to.

I.—AB and CD being given parallel, and being cut by EF in G and H, we have to show why it is impossible that angle AGH should not be equal to angle GHD. Assuming that these angles are not equal, let AGH be the greater. You will remember that in the last proposition we saw that



$\angle$ AGH + $\angle$ BGH = two rt. angles; then since GHD is less than $\angle$ AGH, $\angle$ GHD + $\angle$ BGH are less than two rt. angles; but this is impossible, for by post. 6 we know that when the two interior

angles are less than two right angles, the lines being produced will meet, but AB and CD will not meet, for they are parallel, and since it is impossible that AGH should be unequal to GHD, they must be equal, and these are the alternate angles.

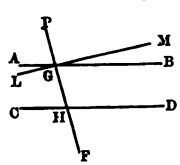
II.—To show that the exterior angle is equal to the interior and opposite. We have just seen that $\angle$ AGH = $\angle$ GHD. In the last prop. we had $\angle$ AGH = $\angle$ EGB (I. 15); therefore the exterior angle EGB is equal to the interior and opposite angle.

III.—To show that the two interior angles on one side of the line are together equal to two right angles

$$\angle \text{ AGH} = \angle \text{ GHD (Pt. 1.)}$$
$$\text{AGH} + \text{BGH} = 2 \text{ rt. angles}$$
$$\therefore \text{GHD} + \text{BGH} = 2 \text{ rt. angles}$$

This concludes the theorem after Euclid's method. The equality of the alternate angles may also be proved by the use of the axiom referred to above, 'That two lines which cut one another cannot both be parallel to the same straight line.'

Assuming as before that $\angle$ AGH is greater than GHD, let us make angle LGH equal to GHD (I. 23), and produce LG to M. Since the alternate angles LGH and GHD are equal, LM is parallel to CD, but AB is parallel to CD, therefore two straight lines which intersect are parallel to the same straight line, which is impossible. The theorem concludes as before.



Again we may adopt Euclid's postulate and alter the arrangement of the theorem, dealing with the two interior angles first, since we are obliged to refer to them in order to show that the alternate angles are equal. The theorem may then be proved in this manner.

I. If two parallels are cut by a third straight line, the two interior angles are together equal to two right angles.

$\angle$ AGH + BGH = 2 rt. angles (I. 13)

$\angle$ CHG + GHD = 2 rt. angles (I. 13)

$\therefore$ all the interior angles = four right angles, and by post. 6 we know that neither of the pairs of interior angles can be less than two right angles; hence neither can be greater, or the four angles would be greater than four right angles. Therefore two interior angles taken on one side of the cutting line must be equal to two right angles.

II. To show that the alternate angles are equal—

$$\text{angles } AGH + BGH = 2 \text{ rt. angles}$$
$$BGH + GHD = 2 \text{ rt. angles}$$

Take away the common angle BGH, and angle AGH = GHD, and they are alternate angles.

III. As above.

SUMMARY OF PROPOSITION XXIX., THEOREM 20.

If two parallels are cut by a straight line :—

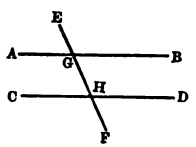
I. The alternate angles are equal.

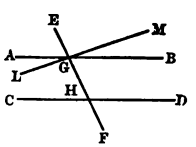
(α, Euclid's post.) Assume $\angle$ AGH greater than $\angle$ GHD.

$\angle$ AGH + BGH = 2 rt. angles (I. 13)

$\therefore$ $\angle$ GHD + BGH are less than 2 rt. angles.

$\therefore$ by post. 6, AB and CD will meet, which is impossible, for they are parallel, &c.





(β, Playfair's post.) Assume $\angle$ AGH greater than $\angle$ GHD, make $\angle$ LGH = $\angle$ GHD (I. 23), and produce LG to M.

LM is $\parallel$ to CD (I. 27), and cuts AB also $\parallel$ to CD, which is impossible, &c.

II. The exterior angle is equal to the interior and opposite.

$\angle$ AGH = $\angle$ GHD (pt. 1)

$\angle$ AGH = $\angle$ EGB (I. 15)

$\therefore$ the exterior $\angle$ EGB = int. and opposite $\angle$ GHD.

III. The two interior angles on one side of the line are equal to two right angles.

$\angle AGH + \angle BGH = 2$ rt. angles (I. 13)

$\angle AGH = GHD$ (pt. 1)

$\therefore \angle GHD + BGH = 2$ rt. angles.

LECTURE XIII.

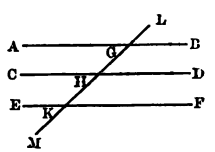
PROPOSITIONS 30–31.—ANALYSIS OF THE THEORY OF PARALLEL STRAIGHT LINES.—PROPOSITION 32 AND COROLLARIES.

Two short and simple propositions, a theorem and a problem, conclude the subject begun in our last lecture.

To show that straight lines which are parallel to the same straight line are parallel to one another.

Let AB and CD be each of them parallel to EF, AB shall be parallel to CD.

In the last proposition, three tests for parallel lines were given us; in order to apply them we must draw a line LM, cutting the given lines in GHK. Then we know that if ∠ AGH = ∠ GHD, *i.e.* if the alternate angles are equal, AB is parallel to CD. Let us see if this is the case.



We know that ∠ AGH = ∠ HKF, because AB is parallel to EF; also we know that the exterior angle GHD is equal to the interior and opposite HKF, because CD is parallel to EF. Then by ax. 1, ∠ AGH = GHD, and it follows that AB is parallel to CD.

Summary of Proposition XXX., Theorem 21.

Straight lines which are parallel to the same straight line are parallel to each other.

Cons.—Draw a line cutting the three given lines.

Proof.—∠ AGH = HKF (alternate ∠s I. 27)

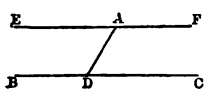
∠ GHD = HKF (ext. = int. and opp. I. 28)

∴ ∠ AGH = ∠ GHD

∴ AB is ∥ to CD (I. 29)

Proposition 31 requires us to draw a straight line through a given point parallel to a given straight line. Let A be the given point, and BC the given straight line. Let us assume that the problem is solved and that a line is drawn through A parallel to BC. What would be the result? Draw a line cutting the parallels, and then from I. 29 it follows that the alternate angles are equal, &c.

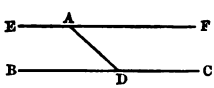
These considerations furnish us with a hint for the construction, for by joining point A with any point D in BC we obtain a line AD, which must cut both parallels when drawn,



and the angle ADC which it makes with BC will be one of the alternate angles, the other of which must evidently be at the point A. By I. 23 we are able to make an angle at this point equal to angle ADC, and producing EA one arm of the angle, through A to any point F, the construction is complete. For EF is parallel to BC, as the alternate angles are equal.

Summary of Proposition XXXI., Problem 10.

Through a given point to draw a straight line parallel to a given straight line.



Cons.—Join AD. Make $\angle$ EAD = $\angle$ ADC (I. 23). Produce EA to F.

Proof.—Alternate angles are equal by cons. $\therefore$ EF is $\parallel$ to BC (I. 27).

This problem concludes a group of five propositions dealing with the theory of parallel straight lines; in the three following theorems we apply the principles here learnt. Before proceeding to these we will sum up the results arrived at.

First you will observe that the test for parallel lines is the equality of certain angles made by a line cutting the supposed parallels.

Lines are parallel if:—

(α) The alternate angles are equal (I. 27).

(β) The exterior angle is equal to the interior and opposite on the same side of the cutting line.

(γ) The two interior angles on the same side are together equal to two right angles (I. 28).

The converse of these three theorems is true.

And if any one of these facts is proved, the other two necessarily follow (I. 29).

We have learnt also to draw a line parallel to another through any given point (I. 31).

Also we may remark that parallel lines often give us a ready means of showing the equality of any angles which we may require for the demonstration of a theorem.

Angles can be shown to be equal, if:—

They form corresponding parts of equal triangles (I. 4, I. 8, I. 26).

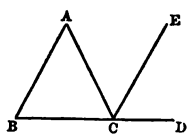
They are opposite equal sides in the same triangle.

They are vertical opposite angles (I. 15).

If two pairs of angles containing a common angle can be shown equal to two right angles, then subtracting the common angle, the remainders are equal.

If they are alternate, or exterior and interior and opposite, made by a line cutting two parallels.

I. 32. We pass to the investigation of Prop. XXXII., which may be considered as the completion of Props. XVI. and XVII. In these we learnt (α) that the exterior angle of a triangle is greater than *either* of the interior and opposite; we now show that it is equal to *both*. (β) That any two angles of a triangle are less than two right angles; we now prove that the *three* interior angles are together equal to two right angles.



I. Let ABC be any triangle, and let BC be produced to D, we have to show that angle ACD is equal to the angles BAC + ABC. We know (I. 17) that angle ACD is greater than BAC or ABC, and we know (I. 23) how to make an angle equal to another; let us then at the point C make an

angle ACE = angle BAC; then, assuming the theorem to be true, the angle ECD, the remaining part of ACD, must be equal to the angle ABC. Let us endeavour to show this.

We know that ∠ BAC = ACE (cons.), and if you will look carefully at the figure you will see that they are alternate angles made by the line AC meeting BA and CE. We may infer that BA and CE are parallel to each other (I. 27), and since that is the case we know that certain other angles as well as the alternate angles are equal. Now, AC makes no other angles with these parallels, but BD also falls upon them and consequently makes the exterior angle ECD equal to the interior and opposite ABC. This gives us the required proof, for the two angles ACE, ECD make up the angle ACD.

In applying this construction to the synthesis of the proposition we can simplify it a little; for, instead of making angle ACE = angle BAC, and then showing that BA and CE are parallel, we may at once draw CE parallel to BA (I. 31), and then it will follow (I. 29) that the alternate angles made by AC are equal, as well as the exterior and its interior opposite angle made by BD.

NOTE.—If in the analysis we had made ∠ ACE = ∠ ABC instead of BAC, the construction would have failed altogether, and it would have been necessary for us to retrace our steps and try the other angle, which, as we have seen, leads to a satisfactory proof.

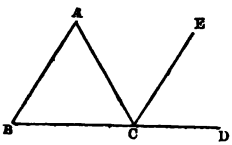
II. We have to show that the three interior angles of triangle ABC are equal to two right angles.

That we may have something with which to compare these angles, let us see if we have any pair of angles in the figure which are equal to two right angles. It is plain that angles ACD, ACB are (I. 13). Now, angle ACB is one of the interior angles. We have just shown that ACD is equal to the other two. Hence it is clear that the three interior angles ABC, BAC, ACB are together equal to two right angles.

SUMMARY OF PROPOSITION XXXII., THEOREM 22.

If a side of any triangle be produced (1) *the exterior angle is equal to the two interior and opposite angles,* (2) *and the three interior angles of every triangle are together equal to two right angles.*

I. To show that the exterior angle = 2 int. and opposite.



Draw CE ∥ to AB.

Proof.—AC falls on ∥s AB and CE, and makes alternate ∠ ACE = ∠ BAC.

BD falls on ∥s, and makes ext. ∠ ECD = int. and opposite ∠ ABC.

∠ ACE + ∠ ECD = ∠ ACD.

II. To show that the three interior angles = 2 rt. angles.

∠ ACD + ∠ ACB = 2 rt. angles (I. 13).

∠ ACD = 2 int. angles of triangle, and ∠ ACB is the third.

Several results are easily deduced from this theorem, which are of great use in solving geometrical problems.

I. If two triangles have two angles of one equal to two angles of the other, their third angles are equal.

For since all the angles of each = 2 rt. angles, and two angles of one = two angles of the other, subtracting equals from equals, we have the remaining third angles equal.

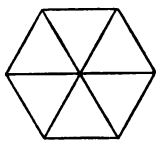
II. In a right-angled triangle the two acute angles are together equal to one right angle. And if the right-angled triangle is isosceles, each of the acute angles contains 45°.

In an equilateral triangle each of the angles is equal to two-thirds of a right angle or 60°.

III. All the interior angles of any rectilineal figure added to four right angles are equal to twice as many right angles as the figure has sides.

Now, any figure can be divided into as many triangles as it has sides by drawing lines from each of its angles to any

point within the figure. And all the angles of these triangles must be equal to twice as many right angles as there are triangles (or sides to the figure). But the point within the figure forms the vertex of all the triangles. All the angles at this point we know from I. 15 are equal to four right angles. Take away these vertical angles and we have remaining the interior angles of the figure, which will clearly be equal to twice as many right angles as the figure has sides, less four right angles.



IV. If we produce all the sides of this figure, we shall have an exterior angle adjacent to every interior angle, and each pair of these angles are equal to two right angles (I. 13). Therefore all the pairs of angles, *i.e.* all the interior and all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides. But all the interior angles + four right angles = twice as many right angles as the figure has sides. Take away the interior angles common to both equations, and we have *all the exterior angles equal to four right angles.*

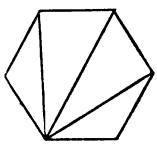
This will be perhaps clearer from an example. Let us take a hexagon or six-sided figure; this will be divided into six triangles, whose angles will be equal to 12 right angles. But the vertical angles = 4 right angles. Therefore the interior angles alone = 8 right angles, or, together with four right angles = 12, *i.e.* twice as many as the hexagon has sides.

Again, producing each side of the hexagon, we have all the exterior and all the interior angles together = 12 right angles—that is, four more than the interior angles alone, whence it is clear that the exterior angles alone = four right angles.

As we have seen, the exterior angles of any rectilineal figure are a constant quantity—that is, no matter what the figure may be, they always amount to four right angles or 360°.

There is another simple method of finding the number of right angles, to which the interior angles of any figure are

equal. If instead of taking a point within the figure, we take one of the angular points, and draw from it straight lines to all the opposite angles, the figure will be resolved into triangles, all of whose angles will be upon the circumference of the figure, and which will therefore be together equal to the interior angles of the figure. And as each triangle contains angles = two right angles, all the interior angles of the figure = twice as many right angles as there are triangles. And it will be found that the number of triangles into which a figure can be thus divided is always two less than the number of sides. Thus, taking the same example as before, a hexagon will be resolved into four triangles, and its interior angles are therefore equal to eight right angles.

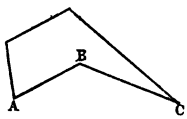


Again we may notice that a hexagon, as we have already seen, resolves itself into four triangles, a pentagon into three, a decagon into eight, &c., and generally that a figure resolves itself into as many triangles as it has sides, less two.

Also taking the various rectilineal figures in order, according to the number of their sides, the triangle is the first in order, since three straight lines are the smallest number that will make a figure; then follow quadrilateral, or four-sided figures, &c., and generally the order of a figure is the same as the number of its sides less two, *i.e.* the same as the number of triangles into which it resolves itself. These facts are often of use in dealing with figures of many sides, *e.g.* a regular quindecagon is a rectilineal figure having fifteen equal sides, it is therefore thirteenth in order, can be resolved into thirteen triangles, and its interior angles are together equal to twenty-six right angles.

Also since its angles are equal, any one of them $= \frac{26 \times 90^\circ}{15}$ $= 156^\circ$. Thus, multiplying the number of angles by 90, and dividing by the number of sides, we can find the size of an angle of any regular polygon.

Lastly, these corollaries do not apply to figures which have what is variously called a reflex, re-entrant, or re-entering angle; thus, angle ABC (interior) is a reflex angle, but, as has been before stated, Euclid does not deal with angles greater than two right angles.



LECTURE XIV.

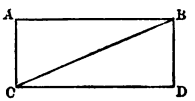
THE PARALLELOGRAM.—PROPOSITIONS 33–34.

I. 33. PROPOSITION XXXIII. shows the development of the parallelogram from parallel straight lines. It teaches us that the straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

Let AB and CD be equal and parallel lines joined by AC and BD, we have to show that AC is equal and parallel to BD.

I. To show that AC is equal to BD.

In order to show that lines are equal, we must generally prove either that they are radii of the same circle or sides of equal triangles. The figure ABDC can be divided into two triangles by joining any two of the angular points. Let us join BC. Then in the triangles ABC, BCD, we have AB = CD by hypothesis and CB common. But do we know anything about the included angle? Since AB and CD are parallels and BC meets them, the alternate angles are equal, viz. ∠ ABC = ∠ BCD.



Therefore the triangles ABC and BCD are equal in all respects, and AC is equal to BD.

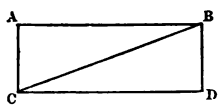
II. It remains to show that AC is also parallel to BD.

AC and BD are met by the line BC, are the alternate angles equal? Yes; for the alternate angles made by BC with AC and BD are also angles of the triangles, which we have just shown to be equal. Therefore AC is parallel to BD.

And the figure ABDC is a parallelogram, that is a four-sided figure whose opposite sides are parallel, and it is formed by joining the extremities of equal and parallel straight lines.

SUMMARY OF PROPOSITION XXXIII., THEOREM 23.

The straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel.



I. To show that AB = CD.

Cons.—Join BC.

Proof.—BC meets ||s AB and CD, and makes

$$\angle ABC = \angle BCD,$$
$$AB, BC = CD, BC;$$
$$\therefore \Delta ABC = \Delta BCD \text{ (I. 4), and } AC = BD.$$

II. To show that AB is ‖ to CD.

BC falls on AC and BD and makes

$$\angle ACB = \angle CBD \text{ (I. 4)};$$
$$\therefore AC \text{ is } \| \text{ to } BD \text{ (I. 29)}.$$

Euclid in the definitions describes four kinds of quadrilateral figures having their opposite sides equal—the square, the rhombus, the rhomboid, and the oblong, more generally spoken of as the rectangle. It may easily be proved that all of these have their opposite sides parallel as well as equal.

Let ABDC be any of these figures, *e.g.* a rhomboid. Join BC. Then the three sides of the two triangles into which it is divided are equal, and the triangles are equal in all respects (I. 8), and hence their angles are equal. Therefore BC falling on AB, CD, and also on AC, BD, in each case makes the alternate angles equal, and therefore each of these pairs of sides are parallels.

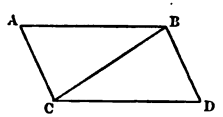
Hence all of these figures may be classed together as parallelograms.

I. 34. In the next proposition we have to show that the

opposite sides and angles of a parallelogram are equal, and that the diameter bisects it.

The diameter, or diagonal, of a parallelogram is the line which joins any pair of its opposite angles, dividing it into two triangles. Let ABDC be a parallelogram, we have to show that AB = CD, AC = BD, and that $\triangle$ ABC = $\triangle$ BCD, which must be the case if the diagonal bisects the figure. It will be better to invert the order in proving the three points of this theorem, for if the triangles are equal it will follow that their sides are equal, and consequently the whole theorem will be proved.

What do we know about these two triangles, as to their sides and as to their angles?



BC is common to the two triangles.

Since AB is parallel to CD, and BC meets them, the alternate angles are equal, *i.e.* $\angle$ ABC = $\angle$ BCD.

Again, since AC is parallel to BD and BC meets them, the alternate angles ACB, CBD are equal.

Then in the two triangles we have two angles and the side between them equal, therefore the triangles are equal in all respects (I. 26, pt. 1).

Since the triangles are equal in area, the diameter bisects the parallelogram.

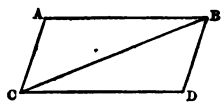
Since the sides of the triangles are equal, each to each, AC = BD, and AB = CD, and the opposite sides of the parallelograms are equal.

Since the angles of the triangles are equal, each to each, $\angle$ CAB = $\angle$ BDC, and adding the other equal angles, $\angle$ ACB + $\angle$ BCD = $\angle$ ABC + $\angle$ CBD, *i.e.* $\angle$ ACD = $\angle$ ABD. Therefore the opposite angles of the parallelogram ABDC are equal.

NOTE.—This is the first proposition in which we have made use of *all* the inferences which can be drawn from the equality of any two triangles; in former propositions we have shown triangles to be equal, in order to prove the equality of certain pairs of angles, or of sides.

Summary of Proposition XXXIV., Theorem 24.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it.



Cons.—Nil.

Proof.—Alternate angles made by BC with both pairs of parallels are equal. BC is common to the two triangles ABC, BCD. $\therefore \triangle ABC = \triangle BCD$ (I. 26). It follows:—

I. AB = CD, AC = BD, $\therefore$ opposite sides of $\|^{gram}$ are equal.

II. $\triangle ABC = BCD$ in area, $\therefore$ diameter bisects $\|^{gram}$ ABDC.

III. $\angle CAB = \angle BDC$;
$\angle ACD\ (ACB + BCD) = \angle ABD\ (ABC + CBD)$
$\therefore$ opposite angles of $\|^{gram}$ are equal.

Let us briefly recapitulate what we have learnt about the parallelogram.

It is formed by joining the ends of equal and parallel straight lines towards the same parts.

Its opposite sides are equal and parallel.

Its opposite angles are equal to each other, all its interior angles are together equal to four right angles and to its exterior angles if the sides be produced.

There are four kinds of parallelograms:—

The square, whose sides and angles are all equal.

The rhombus, whose sides are all equal, but only the opposite angles.

The rhomboid, whose opposite sides and angles are equal.

The oblong or rectangle, whose angles are all equal, but opposite sides only.

Of these figures two, the square and the rectangle, have all their angles right angles.

Two, the rhombus and rhomboid, have two acute and two obtuse angles.

Four-sided figures whose opposite sides are not parallel are called trapeziums. The next proposition (XXXV.) is the only one in Bk. I. in which this figure is referred to.

LECTURE XV.

ON EQUIVALENCE OF AREA.—PROPOSITIONS 35–41.

THE third and last division of Bk. I. shows the conditions under which triangles and parallelograms are equal in area, that is, contain the same amount of space without being necessarily similar in shape (or having equal sides).

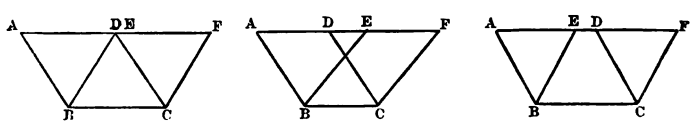
Equality of area is now frequently spoken of as equivalence.

In Proposition XXXV. we are required to show that parallelograms on the same base and between the same parallels are equal in area, or equivalent.

By parallelograms 'on the same base' is meant having a common side, and since the opposite sides of parallelograms are equal, it follows that two sides of one are equal to two sides of the other.

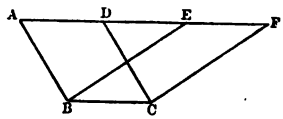
Let ADCB and EFCB be two parallelograms between the same parallels AF, BC, and having a common base BC: we are required to show that ADCB is equal *in area* to EFCB.

In the figure of this proposition we may have three cases; for the sides opposite the base: (1) may meet at a point; (2) may be entirely separate; (3) or may overlap.



The proof of the first case is very easy. Looking at the figure you will see that the triangle DBC is a part of each of the parallelograms; since the diagonal of a parallelogram bisects it, DBC must be a half of each parallelogram, therefore the parallelograms in question must be equal (ax. 6).

Case II. In which AD and EF are separate.



Cover up (with paper or the hand) the parallelogram ADCB, and you will see that a triangle DFC remains. Again, covering the other parallelogram, a similar triangle EAB remains. It is clear, then, that since the whole figure is made up of one of the triangles and one of the parallelograms,

$$\triangle AEB + \parallel^{gram} EFCB = \triangle FDC + \parallel^{gram} ADCB.$$

Therefore if we can show that the triangles are equal, the remainders, the parallelograms, must be equal (ax. 3).

Let us examine these triangles:—

side BE = FC since they are opposite side of $\parallel^{gram}$

side AB = DC for the same reason.

The third sides AE and FD are made up of equal parts AD, FE, opposite to BC, and a common part DE.

Therefore they are equal, and the three sides of the triangles are equal.

Again, because AF meets the parallels DC, AB, the exterior angle FDC = interior and opposite EAB.

Because AF meets EB, FC, angle AEB = DFC.

It follows that we may show that the triangles are equal, by I. 8, I. 4, or I. 26—whichever we please.

Euclid adopts the second method.

Then since the triangles are equal, it follows, as we have shown above, that the parallelograms are.

Case II. The same proof applies if we show that the triangles are equal by I. 26.

If we use I. 4 or I. 8, in sides AE and DF we have to subtract the common part ED instead of adding it as above.

This proof is also applicable to Case I., but that given above is easier. Only one case is given by Euclid.

SUMMARY OF PROPOSITION XXXV., THEOREM 25.

Parallelograms on the same base and between the same parallels are equal in area.

Cons.—Nil.

Case I. Each $\|^{gram}$ is double of Δ DBC (I. 34).

$\therefore$ $\|^{grams}$ are equal.

Cases II. and III. Δ AEB = Δ DFC (I. 4, I. 8, I. 26).

Δ DFC + $\|^{gram}$ ADCB = Δ AEB + $\|^{gram}$ EFCB.

$\therefore$ $\|^{gram}$ ADCB = $\|^{gram}$ EFCB (ax. 3).

(N.B. Equal parts of the triangles.

AE = AD $\pm$ DE, DF = EF $\pm$ DE.

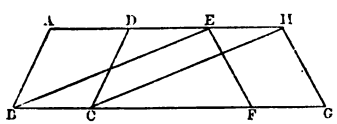
AB = DC, EB = FC, AD and EF = BC (I. 34).

ext. $\angle$ FDC = intr. and opp. $\angle$ EAB.

intr. $\angle$ DFC = ext. and opp. $\angle$ AEB).

Proposition XXXVI. may be considered complementary to the last. It shows that 'Parallelograms on equal bases and between the same parallels are equal.'

Let us draw two parallelograms on equal bases BC and FG, and between the same parallels AH, BG. We have to show that these parallelograms are equal. It is plain that we want a connecting link in order to compare them. We know that parallelograms on the *same* base and between the same parallels are equal; can we construct a parallelogram on BC and between these parallels, which will then be equal to ADCB (I. 35), and try to observe some connection with the other parallelogram? We know that a parallelogram is formed by joining the extremities of equal and parallel straight lines. Is there another line in the figure, other than AD, equal and parallel to BC? EH is equal to FG, and therefore to BC. Then, by joining BE, CH we shall have a third parallelogram, which is equal to ADCB by the last proposition. Also, for the same reason, it is equal to EFGH, for it is on the same base



EH and between the same parallels AH, BG. Therefore, by ax. 1, the two original parallelograms ADCB, EFGH are equal.

SUMMARY OF PROPOSITION XXXVI., THEOREM 26.

Parallelograms on equal bases and between the same parallels are equal in area.

Cons.—Join BE, CH (base of one parallelogram with opposite side of the other).

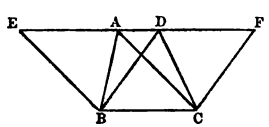
Proof.—EHCB is a $\|^{gram}$ (I. 33).

It is equal to each of the others (I. 35).

$\therefore$ $\|^{gram}$ ADCB = $\|^{gram}$ EFGH.

I. 37. In the next two theorems we prove the corresponding cases of equality of area in triangles. Proposition XXXVII. shows that triangles on the same base and between the same parallels are equal in area.

On the same base BC and between the parallels AD, BC let there be two triangles ABC, DBC. We are required to show that these are equal in area. We know that parallelograms under these conditions are equal; we also know that every parallelogram is divided by its diagonal into two equal triangles. We will therefore consider each of these triangles as half of a parallelogram, and complete these parallelograms. The sides opposite BC can be obtained by producing AD far enough both ways (post. 2).

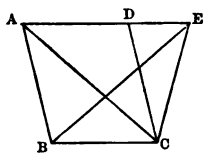


To complete the parallelograms, lines must be drawn through points B and C, parallel to AC and BD respectively (I. 31).

Then EACB and DFCB are parallelograms by construction, and they are on the same base BC. Therefore these are equal in area. Also, since the diagonal bisects them, triangle ABC is half of EACB, and triangle DBC of DFCB; therefore they are equal (ax. 7).

I. 41. From the above it is evident that, if a parallelogram

and a triangle be upon the same base and between the same parallels, the parallelogram is double of the triangle. For each of the parallelograms in the figure is double of one of the triangles (I. 34), and since the triangles are equal, each parallelogram must be double of either triangle. This corollary is enunciated by Euclid as a separate theorem (I. 41). The only construction necessary is to draw the diagonal of the parallelogram in order to obtain a triangle equal to the given triangle, and also half of the parallelogram.



Proposition XXXVIII. is also closely connected with the preceding. Triangles upon equal bases and between the same parallels are equal in area. The construction and proof are precisely similar to that of Proposition XXXVII., except that reference is necessarily made to I. 36, 'parallelograms on equal bases,' instead of to I. 35. We proceed to the summary of these three propositions.

SUMMARY OF PROPOSITIONS XXXVII. AND XXXVIII., THEOREMS 27 AND 28.

Triangles upon the same base, and upon equal bases, and between the same parallels are equal to one another.

Cons.—Make parallelograms of which the triangles are half (I. 31, post. 2).

Proof.—$\|^{\text{grams}}$ on same (I. 35) or equal bases (I. 36) are equal. Triangles are halves of equal $\|^{\text{grams}}$ (I. 34), and are equal (ax. 7).

SUMMARY OF PROPOSITION XLI., THEOREM 31.

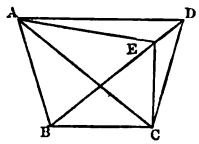
If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.

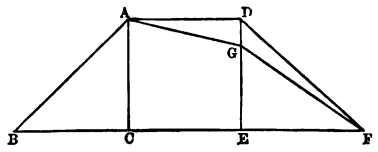
Cons.—Join AC.

Triangles ABC, BEC on same base are equal (I. 37), $\|^{\text{gram}}$ is double of ABC, and therefore of BEC.

Propositions XXXIX. and XL. are the converse of XXXVII. and XXXVIII., and, as is usually the case, are proved by the indirect method. They show that equal triangles on the same base, and on equal bases in the same straight line, and on the same side of it, are between the same parallels. Let us take equal triangles ABC, DBC on the same base BC. Joining the vertices of the triangles, we are required to show that AD is parallel to BC.

Assuming that AD is not parallel to BC, draw AE parallel to BC. Looking at the figure you will see that AE will meet BD (or BD produced), so that by joining EC a triangle EBC will be formed between the same parallels AE, BC, and on the same base BC. Such triangles we know are equal (I. 37), *i.e.* triangle ABC = triangle EBC; but triangle ABC is also equal to DBC, therefore EBC = DBC, the less to the greater, which is impossible. Hence AE is not parallel to BC, and in the same way we can show that no other line than AD can be. Therefore equal triangles on the same base, &c., are between the same parallels. The construction and proof of Prop. XL. are precisely similar, making the reference to I. 38 instead of I. 37.





SUMMARY OF PROPOSITION XXXIX., THEOREM 29.

Equal triangles on the same base and on the same side of it are between the same parallels.

Cons.—Draw AE ‖ to BC. Join EC.

Proof (Indirect).—△ ABC = △ EBC (I. 37).

△ ABC = △ DBC (Hyp.).

∴ △ EBC = △ DBC, less to the greater,

and AE is not ‖ to BC, &c.

SUMMARY OF PROPOSITION XL., THEOREM 30.

Equal triangles on equal bases in the same straight line and on the same side of it are between the same parallels.

Cons. and *Proof* same as above, making reference to I. 38 instead of I. 37.

The results obtained from the propositions considered in this lesson may be briefly summed up.

Parallelograms on the same base, or on equal bases in the same straight line and between the same parallels, are equal in area (I. 35 and 36).

Under similar conditions triangles are equal in area (I. 39 and 40).

And conversely, if equal triangles are on the same base or on equal bases in the same straight line, and on the same side of it, they are between the same parallels.

If a parallelogram and a triangle are on the same base and between the same parallels, the triangle is half of the parallelogram (I. 41).

LECTURE XVI.

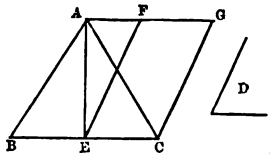
PROPOSITIONS 42–45.

In this lesson, with the exception of one theorem, we shall be engaged in the consideration of some problems the solution of which depends on the principles which we have just learnt. The first is to describe a parallelogram that shall be equal to a given triangle and have one of its angles equal to a given angle. First we must ask what do we know about the relations of parallelograms and triangles in point of area? Prop. XLI. tells us that a parallelogram is double of any triangle on the same base and between the same parallels as itself. Then, if we can divide the given triangle into two equal triangles, we can construct a parallelogram double of either of these (as in I. 37) and consequently equal to the given triangle.

But we have also to make one angle of the parallelogram equal to a given angle; this we can accomplish without difficulty by the use of I. 23, since the size of the angles will not affect the area of the parallelogram. One question remains, how are we to divide the given triangle into two equal triangles? We know from I. 37 and 38 that triangles are equal in area when they are between the same parallels and on the same or on equal bases. Let us endeavour to obtain these conditions.

Let ABC be the given triangle, we have to divide it into two triangles on equal bases and between the same parallels. The equal bases can obviously be obtained by bisecting BC in E (I. 10). Then, if we draw a line through A parallel to BC, AK and BC will be the required parallels. Join EA, and

the required triangles are obtained. We have now to construct the parallelogram on one of the equal bases, say EC. First at the point E (I. 23) in line EC we will make an angle equal to the given angle D, and produce one of its arms to meet the opposite parallel AK in F. To complete the figure we require a side parallel to EF; from point C draw CG parallel to EF (I. 31), and meeting AK in G.



The problem is thus solved, for FECG is by construction a parallelogram (*i.e.* its opposite sides are parallel) having an angle equal to the given angle; also it is double of triangle AEC (I. 41), and therefore equal to triangle ABC.

It follows from this proposition that any triangle may be divided into two others of equal area, by bisecting the base and joining the point of section with the vertical angle.

Summary of Proposition XLII., Problem 11.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Cons.—I. To bisect given triangle.

Bisect BC in E. Join EA. Draw AK ‖ to BC.

II. To construct parallelogram.

At E make ∠ CEF equal to D, let EF cut AK in F (I. 23).

From C draw CG ‖ to EF, cutting AK in G (I. 31).

Proof.—Triangles AEC, AEB on equal bases and between same parallels are equal (I. 38). Therefore AEC = ½ ABC.

FECG is ‖^{gram} having angle = ∠ D by cons.

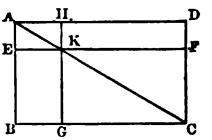
FECG is double of Δ AEC—that is, of half ABC (I. 41).

∴ it is equal to Δ ABC.—Q.E.D.

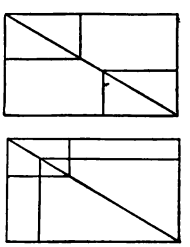
I. 43. We pass on to a theorem which is necessary for the solution of the next problem. 'The complements of the

parallelograms which are about the diameter of any parallelogram are equal to one another.'

The 'parallelograms about the diameter' of any parallelogram are those whose diameters consist of a portion of that of the original parallelogram. The complements are those portions of the figure which, together with the parallelograms about the diameter, make up the whole parallelogram. Draw a parallelogram of which AC is the diameter, then taking any point K in AC draw parallelograms of which AK and KC are diameters; these are parallelograms about AC. Together with these, two small parallelograms, EKGB and HKFD, make up the whole original parallelogram ABCD; these are called the complements, since they complete or fill up the figure, and what the theorem asserts is, that these complements are equal in area. And that this is true you will easily see. For the large parallelogram and 'the parallelograms about the diameter' are each divided by their respective diameters into two triangles, which we know to be equal (I. 34). Also each large triangle is made up of one of each of the smaller triangles together with one complement. Then, taking the smaller triangles which are equal from the large triangles which are equal, it is clear that the remainders, the complements, are equal.



There may be three cases in the figure of this theorem, but the same proof applies to each. For the diameters of the 'parallelograms about the diameter' may be together equal to the diameter of the original parallelogram, as in the example we have taken, (2) or they may only form a part of it, (3) or the parallelograms may overlap so that their diameters are together more than the diameter of the original parallelogram. In the second case the complements are trapeziums, not parallelograms.

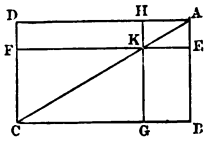


SUMMARY OF PROPOSITION XLIII. THEOREM.

The complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.

Cons.—Nil.

The figure is divided into three equal pairs of triangles (I. 34). One of each of the smaller triangles + one complement = one large triangle.



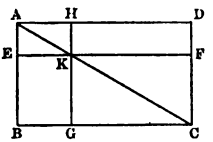
From large triangles take away the smaller triangles, and the remainders, the complements, are equal (ax. 3).

(N.B.—This method of proof is very frequently adopted in Book II.)

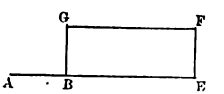
I. 44. A somewhat complicated problem follows: To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle. By the phrase 'apply a parallelogram to a given straight line' is meant that the given line shall form one side of the parallelogram to be applied; the problem might have been enunciated as I. 1 and I. 46; on a given straight line to describe, &c.

You will observe that in the parallelogram required we have fixed the area, one pair of sides, and one pair of angles (since we know that the opposite sides and angles of a parallelogram are equal, I. 33). We can by I. 42 describe a parallelogram fulfilling two of these conditions, *i.e.* equal in area to the given triangle and having an angle equal to the given angle. Draw this parallelogram, calling it EG. The problem now consists in describing a parallelogram on AB equal in area to EG, and also having an angle equal to one of the angles of EG. We must ask under what conditions are parallelograms equal in area? Propositions XXXV. and XXXVI. tell us, when they are on equal or the same bases and between the same parallels, but a moment's consideration will show that we cannot obtain these conditions here. Have we had any other case of equality of area in parallelograms?

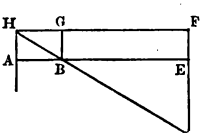
If you will refer to the last proposition you will see that the 'complements' which we proved equal are parallelograms, and that none of the sides of one complement is of necessity equal to one of those of the other, neither are they between the same parallels. And further, if we examine the figure of I. 43, we shall find that it meets all the conditions necessary for the solution of the problem. Let EK represent the line to which the parallelogram is to be applied, and let HF represent the parallelogram equal to the given triangle, and having an angle HKF equal to the given angle. Then EG will be the parallelogram required. For it is applied to line EK, it is equal to HF, the other complement, and it has an angle EKG equal to HKF, since they are opposite vertical angles (I. 15).



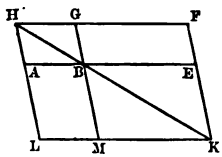
Can we retrace our steps and construct a similar figure? Let us try. Let AB be the given line, C the given triangle, and D the given angle. By I. 42 we can, as we have already remarked, construct a parallelogram EG (which will represent HF in the figure above), *i.e.* equal in area to C, and having an angle at G equal to the given angle D, and we must take care to place EG in such a position that one of its sides shall be in a straight line with AB, and that the angle BGF shall correspond with the angle HKF, thus—



Then AB, BG will be adjacent sides of one of the parallelograms 'about the diameter,' and we can complete it by drawing through A a line parallel to BG, and producing FG to meet it in H. The diameter will be formed by joining HB, and this diameter produced will be the diameter of the large parallelogram, consequently it must meet the opposite side of the large parallelogram which we can obtain by producing FE. But are we sure that HB and FE produced will meet? We know that any straight lines which with another straight line make the two



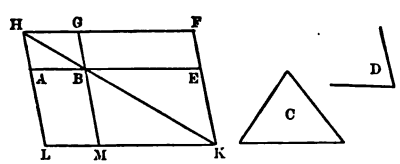
interior angles on the same side together less than two right angles, if produced, will meet on that side (def. 35). We have, then, to ask, are the angles BHF, HFE together less than two right angles? It is evident that the angle BHF is less than the angle AHF of which it is a part, and since AH and FE are parallel, the angles AHF, HFE are equal to two right angles; therefore, the angles BHF, HFE are together less than two right angles, and, consequently, HB and FE will meet if produced towards BE. Let them be produced and meet in K. The triangle HFK thus formed will be half of the large parallelogram which we are trying to construct, and which will be completed by drawing a line through K parallel to HF, and producing HA to meet it in L. To obtain the other complement, which is the required parallelogram applied to AB, we have only to produce GB to meet LK in M.



Summary of Proposition XLIV., Problem 12.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.

Cons.—Make ‖gram BGFE equal to given triangle, and having ∠ EBG equal to given angle. Place it so that BE, one of the arms of ∠ EBG, is in the same straight line with AB (I. 42). (One complement.) Through A draw a line parallel to BG, and produce FG to meet it in H (I. 31.) Join HB. (One ‖gram about diameter.) HB and FE will meet if produced, since angles BHF, HFE are less than two right angles. Let them meet in K. Through K draw a line parallel to HF, and produce HA to meet it in L. (Large parallelogram.) Produce GB to meet KL in M. (Remaining ‖gram and complement.)



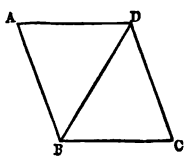
Proof.—Comps. BL and BF of parallelogram HLKF are equal.

$\angle$ ABM = $\angle$ GBE (I. 15) = given angle.

BL = BF = Δ C.

Therefore to AB a $\|^{gram}$ BL has been applied, equal to Δ C, and having an angle ABM equal to $\angle$ D.—Q.E.F.

I. 45. The next problem is to describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle. In Proposition XXXII. we learnt how to divide any rectilineal figure into triangles by drawing lines from one of its angles to all of the opposite angles. The figure given in the problem has four sides, and therefore can be divided into two triangles by joining either pair of opposite angles. Let ABCD be the given rectilineal figure thus divided by BD, and let E be the given angle.



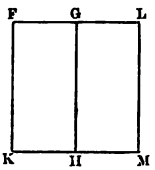
Now by I. 42 we can construct a parallelogram equal to one of these triangles, and by the last proposition we can apply to one side of this parallelogram another, equal to the remaining triangle, and in each case we can make one of the angles equal to the given angle E. And since we wish the whole figure to form a parallelogram we must take the angle equal to E in the second parallelogram opposite to the corresponding angle in the first, thus: Let FGHK be the parallelogram constructed equal to triangle ABD, and having its angle FKH equal to angle E (I. 42), and to its side GH apply a parallelogram GLMH equal to triangle DCB, and having angle GLM equal to angle E.

Then FKML is a figure consisting of two parallelograms which are together equal to the given figure ABCD, and it has an angle equal to the given angle E. It remains to show that it is a parallelogram, *i.e.* that its opposite sides are parallel.

I. The sides FK and LM are each of them parallel to GH

(a common side to the two parallelograms). Therefore, they are parallel to each other.

Again, FG, GL are parallel to KH, HM, each to each. But are these parts in the same straight line so that they form a side of the parallelogram? What test have we had for showing when two straight lines are 'in one and the same straight line'? We must refer to Proposition XIV., which tells us that 'when a straight line meeting two other straight lines makes the adjacent angles equal to two right angles, these straight lines are in one and the same straight line.'



(1) Are angles GHK, GHM equal to two right angles? If so KHM is a straight line.

(2) Also, are angles FGH, LGH equal to two right angles? If so FGL is a straight line.

Then, examining the angles, we know that angles FKH, GHK are equal to two right angles, since GH is parallel to FK (I. 29), and angle GHM = angle GLM = angle FKH. Therefore, the angles GHK, GHM are together equal to two right angles, and KHM is a straight line.

Similarly angles HGL, GLM are equal to two right angles (I. 29). But angle FGH = ∠ FKH = ∠ GLM (I. 33). Therefore, angles FGH, HGL = two right angles, and FGL is a straight line, and it is parallel to KHM. It follows that the figure FKML is a parallelogram, and we have shown that it is equal to ABCD, and that it has an angle FKM equal to E.

In the same way a parallelogram may be constructed equal to a rectilineal figure of any number of sides, by resolving the figure into triangles, then by I. 42 drawing a parallelogram equal to one of the triangles, and applying successively parallelograms equal to the other triangles.

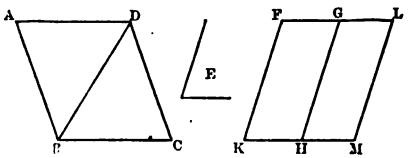
NOTE.—In showing that FGL is a straight line Euclid reasons thus: The angles MHG, HGL are equal to two right angles (I. 29), and the alternate angles FGH, GHM are equal; therefore the angles FGH, LGH are equal to two right angles, and FGL is a straight line, and it is parallel to KHM.

SUMMARY OF PROPOSITION XLV., PROBLEM 13.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.

Cons.—Divide given figure into triangles by joining BD.

Make a parallelogram equal to one triangle, and having angle FKH equal to given angle (I. 42).



Apply to side GH a ‖gram equal to the other triangle, and having an angle GHM or GLM equal to E (I. 44).

FKML is the ‖gram required.

Proof.—FK and LM are both ‖ to GH.

∴ FK is ‖ to LM (I. 30).

∠ GHM = ∠ FKH (cons.). Add ∠ KHG.

∠s GHM, KHG = ∠s FKH, KHG = 2 rt. angles (I. 29).

∴ KHM is a straight line (I. 14).

∠ FGH = ∠ FKH = ∠ GLM. Add ∠ HGL.

∠s FGH, HGL = ∠s HGL, GLM = 2 rt. angles (I. 29).

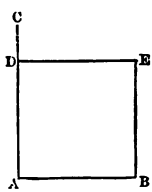
∴ FGL is a straight line.

And since the parts of FL and KM are ‖ (cons.), the whole lines are ‖, and FKLM is a ‖gram by construction, and it was made equal to the two parts of ABCD, and has an angle FKM equal to E.—Q.E.F.

LECTURE XVII.

THE SQUARE.—PROPOSITIONS 46–48.—ANALYSIS OF SECTION 3 OF BOOK I.

THE three concluding propositions of Book I. deal with the square. Proposition XLVI. requires us to describe a square on a given straight line AB. From def. 30 we learn that a square is a four-sided figure, having all its sides equal, and all its angles right angles. It is evident from this definition that, to begin with, we must make a right angle at A or B. How can we do this? Proposition XI. teaches us to draw a line at right angles to another from a given point in the same. From point A draw AC at right angles to AB; the angle BAC will form one of the angles of the square, and by cutting



off from AC (produced if necessary) AD equal to AB we shall obtain another side. Since a square is also a parallelogram, the two remaining sides must be respectively parallel to AD, AB, already drawn. Draw through points B, D lines parallel to AD and AB, and meeting at a point E. Are the sides of this figure equal? ADEB is a parallelogram by construction, and its opposite sides are equal; therefore DE = AB, EB = AD, and AD was made equal to AB; consequently, the four sides are equal (ax. 1).

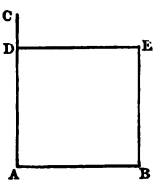
Are the angles all right angles? ∠ BAD is a right angle by construction; therefore, the opposite angle DEB is a right angle (I. 33). Again, because ED and AB are parallel, and AD meets them, the two interior angles EDA, DAB are equal to two right angles (I. 29), and BAD is a right angle (cons.); therefore, EDA is a right angle, and the opposite

angle ABE is a right angle (I. 33). That is, all the angles of ADEB are right angles, and all its sides are equal; therefore, it is a square, and it is described on the given line AB.

Summary of Proposition XLVI., Problem 14.

On a given straight line to describe a square.

Cons.—From A draw AC at right angles to AB; cut off AD = AB.



Through B and D draw BE and DE ‖ to AD, AB.

Proof.—ADEB is ‖^{gram} (cons.); ∴ DE = AB, EB = AD.

AD = AB (cons.); ∴ ADEB is equilateral.

∠s BAD, ADE = 2 rt. angles (I. 29).

BAD is a rt. angle (cons.); ∴ ∠ ADE is also.

Opposite angles of a ‖^{gram} are equal; ∴ remaining angles are rt. angles.

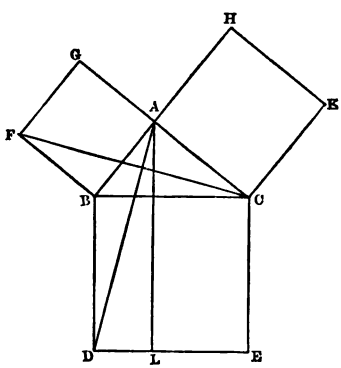
∴ ADEB is a square, and it is described on AB.—Q.E.F.

Proposition XLVII. shows that, supposing squares to be described on all the three sides of a right-angled triangle, the squares on the sides containing the right angle are together equal in area to the square on the side opposite to the right angle. This side is frequently spoken of as the hypothenuse (*ὑποτενοῦσα* stretched under, *i.e.* the line stretched under the angle). Let us take any right-angled triangle BAC, and let BAC be the right angle. Having described squares on all three sides, we are required to show that the squares on BA, AC are together equal to the square on BC.

We may imagine that this theorem was discovered somewhat in the manner following.

First, assuming the theorem to be true, it must be possible to divide the square on BC into two parallelograms, which will be respectively equal to the squares on BA and AC. By I. 45 we can apply to one of the sides of the square on

BC, say BD, a parallelogram equal to one of the squares on the sides containing the right angle, say BA, and having one of its angles equal to BDE, *i.e.* a right angle. This parallelogram, which we will call BL, will, from its construction, coincide with a part of the square on BC, and the remaining part, the parallelogram LC, must be equal in area to the square on AC. Now, if you will look at the figure thus obtained, you will notice that the line defining the two ‖grams will, if produced, pass through the point A. And you will find that this is the case in any right-angled triangle that can be drawn, providing that the constructed parallelogram be applied to that side of the square on the hypothenuse which is adjacent to the square to which the applied parallelogram is equal. This fact gives us the clue to a part of the construction required, for if the line which divides the square on BC into parallelograms equal to the squares on BA, AC always passes through A, it is easy to construct the parallelograms required, by drawing through A a line parallel to BD or CE cutting DE in L.



Now, adopting this construction, how can we show that BL is equal to the square on BA? We know that parallelograms are equal under certain conditions when they are between the same parallels, but these are not between the same parallels. We know also that parallelograms which are the complements of a larger parallelogram are equal, but these are not. What other test of equality have we, then? We have no direct test left, but we know that a parallelogram is double of a triangle on the same base and between the same parallels (I. 41). Then, if we can show that the square on BA, and BL are double of the same or equal triangles, the proof will be complete. Can

such a triangle or triangles be constructed? I think you will easily see that by joining AD we shall obtain a triangle ABD between the same parallels AL and BD, on same base BD as the parallelogram BL, and of which BL is therefore double.

But the square on BA and triangle ABD are obviously not between the same parallels, and we must endeavour then to construct a triangle similar to ABD and within the same parallels as the square on BA. Now triangle ABD is formed by a side of the square on BA and a side of the square on BC and the line AD joining their extremities. Then taking FB, BC and joining FC we shall obtain a triangle similar to ABD. About this triangle we must answer two questions. (1) Is it half of the square on BA? It is on the same base BF, and between the same parallels, for GA, AC are in the same straight line, since angles GAB, CAB are both right angles (I. 14). Therefore the square on AB is double of the triangle FBC (I. 41). (2) Is it equal to the triangle ABD? sides FB, BA are equal, since they are sides of square on BA
sides BC, BD are equal „ „ „ „ „ BC

Angles ABD, FBC are each made up of a right angle + angle ABC, and are therefore equal; and the triangle ABD is equal to the triangle FBC (I. 4). And since the parallelogram BL, and the square on BA, are respectively double of these triangles, they are equal.

In the same way by joining KB and AE we can show that the parallelogram CL is equal to the square on AC. Therefore the squares on AB, AC are together equal to the square on BC.—Q.E.D.

We proceed to sum up and arrange the synthesis of the proposition.

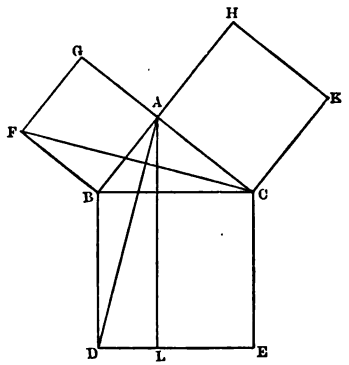
Summary of Proposition XLVII., Theorem 33.

In any right-angled triangle the square on the hypothenuse is equal to the squares on the sides containing the right angle.

Cons.—The squares on the three sides of triangle ABC being drawn,

Through A draw AL ‖ to BD or CE and meeting BD (I. 31).

Join AD, FC.



Proof.— $\triangle ABD = \triangle FBC$ (I. 4)

$\left(\begin{matrix}\angle FBC = \\ \angle ABD = \end{matrix}\right\}$ a rt. angle + $\angle ABC$)

sq. on BA is double of $\triangle$ FBC.

$\|^{gram}$ BL is double of $\triangle$ ABD.

$\therefore$ sq. on BA = $\|^{gram}$ BL.

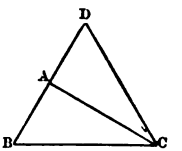
In same manner show that LC = sq. on AC, &c.

The discovery of this theorem is ascribed to Pythagoras; it is said that he was so delighted at his success that he sacrificed an ox in honour of the occasion.

I. 48. The book concludes with the converse of this proposition, which, however, is not demonstrated indirectly, as is so frequently the case. In his proof Euclid makes use of two facts which are evident from the nature of the square, but to which he has not previously called attention: (1) that squares on equal straight lines are equal, and (2) conversely that equal squares must be on equal straight lines. The theorem to be proved is thus enunciated: if the square described on one of

the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let BAC be a triangle such that the squares on BA, AC are together equal to the square on BC, it follows that the angle BAC is a right angle.



Euclid proves this by constructing on AC a triangle similar to BAC. This is done by drawing a line at right angles to AC, making a part AD equal to AB and joining DC.

Then since angle DAC is a right angle by construction, squares on DA, AC = square on DC.

And since DA = AB (cons.), squares on DA, AC = squares on BA, AC = square on BC (hyp.).

Therefore the square on DC = square on BC, and line DC = line BC.

Then by I. 8 the triangles DAC and BAC are equal, and the angle BAC = angle DAC.

But DAC is a right angle (cons.), therefore the angle BAC is a right angle.—Q.E.D.

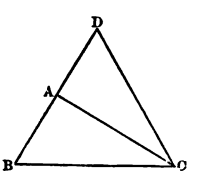
Summary of Proposition XLVIII., Theorem 34.

If the squares on the sides containing an angle of a triangle are together equal to the square on the line subtending it, the angle in question is a right angle.

Cons.—Draw a line at right angles to AC, make AD = AB.

Join DC.

Proof.—Sqq. DA, AC = sqq. on BA, AC
= sq. on BC
but sqq. on DA, AC = sq. on DC
∴ sq. on BC = sq. on DC
and BC = DC.



Then BA = DA, and AC is common
∴ ∠ BAC = ∠ DAC (I. 8) = rt. angle.

The third section of Book I. consists of:—

I. Eight theorems giving the conditions of equivalence of area in triangles and parallelograms (Props. 35–41, and 43). The results of these have been already tabulated.

II. Three problems applying these conditions, viz. to describe a parallelogram equal (α) to a given triangle, with or without one side being given, (β) to any given rectilineal figure. In all three problems one angle of the parallelogram required is to be equal to a given angle. (I. 42, 44, 45.)

III. Three propositions, one problem, and two theorems relating to the square.

On a given straight line to describe a square (I. 46).

The square on the hypothenuse is equal to the squares on the sides containing the right angle, in any right-angled triangle, and conversely (I. 47, I. 48).

EUCLID'S ELEMENTS.

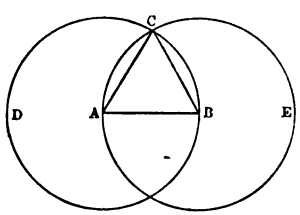
BOOK I.

Proposition I., Problem 1.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line:

It is required to describe an equilat. Δ on AB.



From the centre A, at the distance AB, describe the circle BCD. (Post. 3.)

From the centre B, at the distance BA, describe the circle ACE. (Post. 3.)

To the point C at which the circles cut one another, from points A and B draw the straight lines AC, BC. (Post. 1.)

ABC shall be an equilateral triangle.

For $\because$ A is the centre of the circle BCD, AC = AB

And $\because$ B is the centre of the circle ACE, BC = AB

$\therefore$ AC, BC each = AB.

But things which are equal to the same thing are equal to one another (ax. 1).

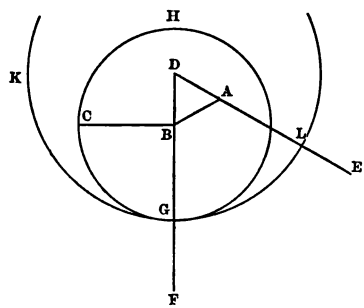
$\therefore$ AC = BC.

Thus AC, AB, BC are all equal, and an equilat. Δ ABC has been described on AB.—Q.E.F.

Proposition II., Problem 2.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw from A a straight line equal to BC.



Join A, B. (Post. 1.)

On AB describe the equilateral △ DAB (I. 1).

Produce the straight lines DA, DB to E and F. (Post. 2.)

From the centre B, at distance BC, describe the circle CGH meeting DF in G. (Post. 3.)

From the centre D, at the distance DG, describe the circle GKL meeting DE in L.

AL shall be equal to BC

For ∵ B is the centre of circle CGH, BC = BG

And ∵ D is the centre of circle GKL, DL = DG. (def. 15).

Also parts of these DA and DB are equal (def. 24).

∴ remainder AL = remainder BG (ax. 3).

But BC = BG; ∴ AL = BC.

Thus from point A a straight line AL has been drawn= BC.—Q.E.F.

Proposition III., Problem 3.

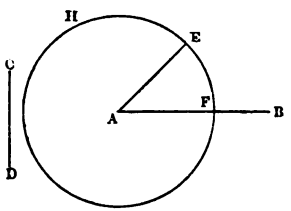
From the greater of two given straight lines to cut off a part equal to the less.

Let AB and CD be the two given straight lines, of which AB is the greater.

It is required to cut off from AB a part = CD.

From point A draw the straight line AE = CD (I. 2).

From centre A, at distance AE, describe circle EFH cutting AB in F.



AF shall be equal to CD.

For ∵ A is the centre of the circle EFH : AF = AE.

But AE = CD (cons.);

∴ AF = CD (ax. 1).

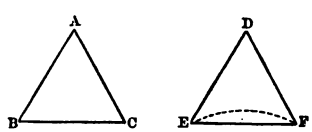
Thus from AB a part AF has been cut off equal to CD.—Q.E.F.

Proposition IV., Theorem 1.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another: then they shall also have their bases or third sides equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite.

In the △s ABC, DEF, let AB = DE, and AC = DF and ∠BAC = ∠EDF.

Then shall BC = EF and △ ABC = △ DEF and ∠ABC = ∠DEF and ∠ACB = DFE.



For if △ ABC be applied to △ DEF so that the point A may be upon D and the straight line AB on the straight line DE,

then ∵ AB = DE, the point B will coincide with point E. And AB coinciding with DE, AC will fall on DF, ∵ ∠BAC = ∠EDF.

And ∵ AC = DF, the point C will coincide with point F.

But point B was shown to coincide with point E,

∴ the base BC will coincide with the base EF; if not, let it fall otherwise as EOF; then the two straight lines BC, EF will enclose a space, which is impossible. (Post. 5.)

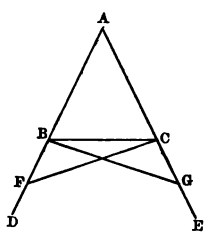
$\therefore$ BC will coincide with EF and is equal to it (ax. 8),
and $\triangle$ ABC will coincide with $\triangle$ DEF and is equal to it,
and $\angle$ ABC „ „ $\angle$ DEF „ „
and $\angle$ ACB „ „ $\angle$ DFE „ „
Wherefore if two triangles &c.—Q.E.D.

Proposition V., Theorem 2.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

Let ABC be an isosceles $\triangle$, having side AB = side AC, and let AB and AC be produced to D and E:

$\angle$ ABC shall be equal to $\angle$ ACB;
and $\angle$ DBC „ „ $\angle$ ECB.



In BD take any point F.
From AE cut off AG = AF.
Join FC and GB.
Then in $\triangle$s AFC, AGB, FA = GA,
and AC = AB, and $\angle$ FAC = $\angle$ GAB;
$\therefore$ base FC = base GB, and $\triangle$ AFC
= $\triangle$ AGB (I. 4),
and $\angle$ ACF = $\angle$ ABG and $\angle$ AFC
= $\angle$ AGB.

Also $\because$ whole AF = whole AG, of which the parts AB, AC are equal,

$\therefore$ remainder BF = remainder CG (ax. 3).

Then in $\triangle$s BFC, GCB,

BF = CG, and FC = GB, and $\angle$ BFC = $\angle$ CGB;

$\therefore$ $\triangle$ BFC = $\triangle$ CGB (I. 4),

and $\angle$ FBC = $\angle$ GCB, and $\angle$ BCF = $\angle$ CBG (I. 4).

And it has been shown that $\angle$ ABG = $\angle$ ACF, and that parts of these, $\angle$ CBG and $\angle$ BCF are equal.

$\therefore$ remaining $\angle$ ABC = remaining $\angle$ ACB, which are the angles at the base of $\triangle$ ABC.

Also it has been shown that $\angle$ FBC = $\angle$ GCB, which are the angles on the other side of the base.

Wherefore the angles, &c.—Q.E.D.

Corollary.—Hence every equilateral triangle is also equiangular.

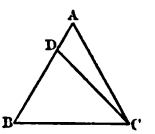
PROPOSITION VI., THEOREM 3.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to the equal angles, shall be equal to one another.

Let ABC be a Δ, having $\angle$ ABC = $\angle$ ACB; the side AC shall be equal to the side AB.

For if AC be not equal to AB one of them must be greater than the other.

Let AB be the greater, and from it cut off DB = AC the less (I. 3). Join DC.



Then in Δs ABC, DBC,

AC = DB (cons.), and BC is common,

and $\angle$ ACB = $\angle$ DBC (hyp.).

$\therefore$ Δ ABC = Δ DBC, the less to the greater, which is impossible (ax. 9).

$\therefore$ AB is not unequal to AC, that is, it is equal to it.

Wherefore if two angles, &c.—Q.E.D.

Corollary.—Hence every equiangular triangle is also equilateral.

PROPOSITION VII., THEOREM 4.

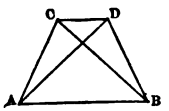
On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

If it be possible on the same base AB, and on the same side of it, let there be two Δs ACB, ADB, having AC and AD terminated at A equal, and *also* CB, DB terminated at B equal to one another.

Join CD.

Case I.—When the vertex of each triangle is without the other.

$\therefore$ AC = AD, $\angle$ ACD = $\angle$ ADC (I. 5).



But $\angle$ ACD is greater than $\angle$ BCD (ax. 9),

$\therefore$ $\angle$ ADC is also greater than $\angle$ BCD; much more then is $\angle$ BDC greater than $\angle$ BCD.

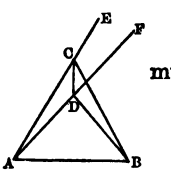
Again $\because$ BC = BD, $\angle$ BDC = $\angle$ BCD.

But it has been shown to be greater, which is impossible.

Case II.—When one vertex D is within the other triangle.

Produce AC, AD to E and F.

Then $\because$ AC = AD, in Δ ACD (hyp.), $\angle$s ECD, FDC, on the other side of the base are equal (I. 5).



But $\angle$ ECD is greater than $\angle$ BCD;

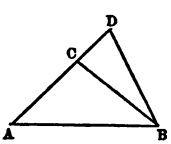
$\therefore$ $\angle$ FDC „ „ $\angle$ BCD,

much more then $\angle$ BDC „ „ $\angle$ BCD.

Again $\because$ BC = BD in Δ BDC,

$\angle$ BDC = $\angle$ BCD.

But it has been shown to be greater, which is impossible.



The case in which the vertex of one Δ is on a side of the other needs no demonstration.

Wherefore on the same base, &c.—

Q.E.D.

PROPOSITION VIII., THEOREM 5.

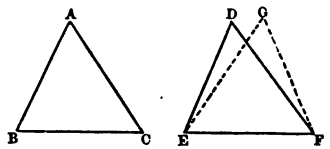
If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides equal to them of the other.

Let the sides of the Δs ABC, DEF be equal, each to each: AB = DE, AC = DF, and BC = EF; the $\angle$ BAC shall be equal to $\angle$ EDF.

For if the Δ ABC be applied to Δ DEF, so that point B may be upon point E, and the straight line BC on EF, the point C will coincide with point F, $\because$ BC = EF.

∴ BC coinciding with EF, BA and AC will coincide with ED and DF.

For if the base BC coincides with the base EF, but the sides BA, AC do not coincide, but have a different situation, as EG, GF, then on the same base and on the same side of it there will be two triangles having their sides which are terminated at one extremity of the base equal, and likewise those which are terminated at the other extremity. But this is impossible (I. 7).



∴ since base BC coincides with base EF, the sides BA, AC must coincide with the sides ED, DF.

∴ ∠ BAC coincides with ∠ EDF, and is equal to it.

Wherefore if two triangles &c.—Q.E.D.

Proposition IX., Problem 4.

To bisect a given rectilineal angle.

Let BAC be the given rectilineal angle:

It is required to bisect it.

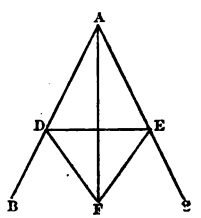
Take any point D in AB.

From AC cut off AE = AD (I. 3);

Join DE, and on the side of DE remote from A, describe the equilat. Δ DEF (I. 1).

Join AF.

The straight line AF shall bisect the given angle BAC.



Then in the Δs DAF, EAF,

DA = AE,

AF is common,

the base DF = base EF (def. 24).

∴ the ∠ DAF = ∠ EAF (I. 8).

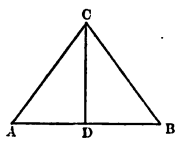
Wherefore the given angle BAC is bisected by the straight line AF.—Q.E.F.

Proposition X., Problem 5.

To bisect a given finite straight line.

Let AB be the given straight line:

It is required to divide it into two equal parts.



On AB describe an equilat. Δ ABC (I. 1).

Bisect $\angle$ ACB by the straight line CD, meeting AB in D (I. 9).

AB shall be bisected at the point D.

In the Δs ACD, BCD

AC = CB (I. 24),

CD is common to the two Δs,

$\angle$ ACD = $\angle$ BCD (cons.);

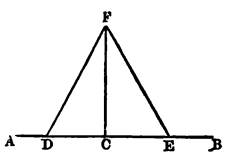
$\therefore$ base AD = base DB (I. 4).

Wherefore the given straight line AB is bisected at the point D.—Q.E.F.

Proposition XI., Problem 6.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line and C the given point in it.



It is required to draw from C a straight line at right angles to AB.

Take any point D in AC, from CB cut off CE = CD.

On DE describe the equilat. Δ DFE.

Join CF.

The straight line CF drawn from point C shall be at right angles to AB.

In the Δs DCF, ECF,

DF = FE (def. 24),

DC = CE (cons.),

FC is common to the two Δs.

$\therefore$ $\angle$ DCF = $\angle$ ECF (I. 8).

But when a straight line falling upon another straight line makes the adjacent angles equal, each of the angles is a right angle (def. 10).

∴ each of the angles DCF, ECF is a right angle.

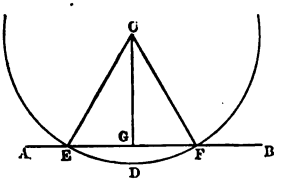
Wherefore from the given point C in AB, CF has been drawn at right angles to AB.—Q.E.D.

Proposition XII., Problem 7.

To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.

Let AB be the given line of unlimited length, and C the given point without it.

It is required to draw from point C a perpendicular to AB.



Take any point D on the other side of AB.

From centre C, at distance CD, describe a circle cutting AB in E and F.

Bisect EF in G (I. 10).

Join CE, CG, CF.

CG shall be perpendicular to AB.

In the △s EGC, FGC,

EC = CF (def. 15),

EG = GF (cons.),

GC is common to the two △s.

∴ ∠ EGC = ∠ CGF, and they are adjacent angles.

∴ they are right angles, and CG is perpendicular to AB.

Wherefore from point C, &c.—Q.E.D.

Proposition XIII., Theorem 6.

The angles which one straight line makes with another, on one side of it, either are, or are together equal to, two right angles.

Let the straight line AB make with the straight line CD, on one side of it, the ∠s CBA, ABD.

These either are, or are together equal to, two rt. ∠s.

For if ∠ CBA = ∠ DBA, each of them is a rt. ∠ (def. 10).

But if not, from point B draw BE at right angles to CD (I. 11).

Then the ∠s CBE, EBD = two rt. angles.

Now, the ∠ CBE = the two ∠s CBA, ABE, to each of these equals add the angle EBD.

∴ the ∠s CBE, EBD = the three angles CBA, ABE, EBD (ax. 2).

Again, ∠ DBA = the two ∠s DBE, EBA, to each of these equals add ∠ ABC.

∴ the angles DBA, ABC = the same three ∠s DBE, EBA, ABC (ax. 3).

∴ the ∠s DBA, ABC = ∠s CBE, EBD = two rt. ∠s.

Wherefore the angles, &c.—Q.E.D.

Proposition XIV., Theorem 7.

If at a point in a straight line two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At point B, in the straight line AB, let the two straight lines BC, BD on the opposite sides of AB make the adjacent ∠s ABC, ABD together equal to two rt. ∠s; BD shall be in the same straight line with CB.

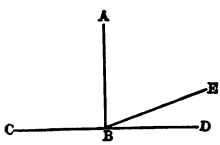
For if not, let BE be in the same straight line with BC.

Then ∠s ABC, ABE together = two rt. ∠s (I. 13),

and ∠s ABC, ABD „ „ „ (Hyp.),

∴ ∠s ABC, ABE „ „ ∠s ABC, ABD.

From these equals take away the common angle ABC.



Then ∠ ABE = ∠ ABD, the less to the greater, which is impossible.

∴ BE is not in the same straight line with BC, and in the same manner it can be shown that no other line but BD can be.

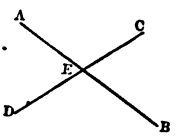
Wherefore if at a point, &c.—Q.E.D.

Proposition XV., Theorem 8.

If two straight lines cut one another, the vertical or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another at point E.

The ∠ AEC shall be equal to ∠ DEB, and the ∠ CEB to the ∠ AED.



For ∵ AE meets CD, ∠s AEC, AED together = two rt. ∠s (I. 13).

And ∵ DE meets AB, ∠s BED, AED together = two rt. ∠s (I. 13).

∴ ∠s AEC, AED together = ∠s BED, AED.

From these equals take away the common ∠ AED, and the remaining ∠ CEA = remaining ∠ DEB.

In the same manner it may be shown that the ∠ CEB = ∠ AED.

Wherefore if two straight lines, &c.—Q.E.D.

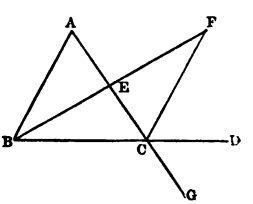
Corollary.—1. Hence it is manifest that if two straight lines cut one another the angles which they make at the point of section are together equal to four right angles. Cor. 2. And, consequently, that all the angles made by any number of straight lines meeting at one point are together equal to four right angles.

Proposition XVI., Theorem 9.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let BC, a side of △ ABC, be produced to D.

The ∠ ACD shall be greater than either ∠ CAB or ∠ ABC.



Bisect AC in E. Join BE.

Produce BE to F, making EF = BE.

Then in △s BEA, FEC,

BE = FE.

EA = EC.

$\angle$ BEA = $\angle$ FEC $\because$ they are opposite vertical $\angle$s (I. 15).

$\therefore$ $\angle$ ECF = $\angle$ EAB.

But $\angle$ ACD is greater than $\angle$ ECF (ax. 9).

$\therefore$ $\angle$ ACD is greater than $\angle$ EAB, *i.e.* $\angle$ CAB.

In the same manner, if AC be produced to G it may be shown that $\angle$ BCG is greater than $\angle$ ABC.

And $\angle$ BCG = $\angle$ ACD.

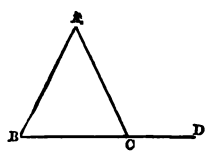
$\therefore$ $\angle$ ACD is greater than $\angle$ ABC.

Wherefore if one side, &c.—Q.E.D.

Proposition XVII., Theorem 10.

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle.



Any two of its angles are together less than two rt. $\angle$s.

Produce BC to D.

Then $\because$ $\angle$ ACD is the exterior $\angle$ of Δ ABC.

$\angle$ ACD is greater than the interior and opposite $\angle$ ABC.

To each of these add $\angle$ ACB.

Then $\angle$s ACB, ACD are together greater than $\angle$s ACB, ABC.

But $\angle$s ACB, ACD together = two rt. $\angle$s (I. 13).

$\therefore$ $\angle$ ACB, ABC are together less than two rt. $\angle$s.

In the same manner it may be shown that the angles BAC, ACB, as also $\angle$s CAB, ABC are together less than two rt. $\angle$s.

Wherefore any two angles, &c.—Q.E.D.

Proposition XVIII., Theorem 11.

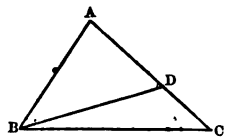
The greater side of every triangle has the greater angle opposite to it.

In Δ ABC let the side AC be greater than the side AB.

The $\angle$ ABC shall be greater than the $\angle$ ACB.

From AC the greater cut off AD = AB. Join BD.

Then ∵ ∠ ADB is the exterior ∠ of △CDB, it is greater than the interior opposite ∠ DCB (I. 16).



But ∵ AB = AD, ∠ ADB = ∠ ABD (I. 5).

∴ ∠ ABD is greater than ∠ ACB.

Much more then is ∠ ABC greater than ∠ ACB (ax. 9).

Wherefore the greater side, &c.—Q.E.D.

Proposition XIX., Theorem 12.

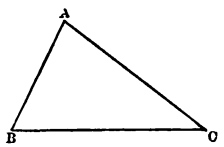
The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

In △ABC let ∠ABC be greater than ∠ACB:

The side AC shall be greater than the side AB.

For if not, AC must be either equal to or less than AB.

But AC is not equal to AB, for then ∠ABC would = ∠ACB (I. 5), but it is not (hyp.).



∴ AC is not equal to AB.

Neither is AC less than AB, for then ∠ABC would be less than ∠ACB (I. 18), but it is not (hyp.).

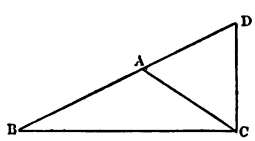
∴ AC is greater than AB.

Wherefore the greater angle, &c.—Q.E.D.

Proposition XX., Theorem 13.

Any two sides of a triangle are together greater than the third side.

Let ABC be a △; any two of its sides are together greater than the third side.



Produce BA to D, making AD = AC. Join DC.

Then ∵ AD = AC (cons.)

$\angle ADC = \angle ACD$ (I. 5).

But $\angle BCD$ is greater than $\angle ACD$ (ax. 9.)

$\therefore$ $\angle BCD$ is greater than $\angle BDC$.

$\therefore$ in $\triangle DBC$, the side BD is greater than BC (I. 19). But BD = BA and AC.

$\therefore$ BA and AC are greater than BC.

In the same manner it may be shown that AB, BC are greater than AC, and BC, CA greater than AB.

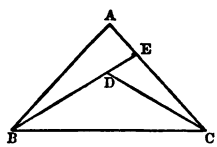
Wherefore any two sides, &c.—Q.E.D.

Proposition XXI., Theorem 14.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a $\triangle$, and from D a point within it draw straight lines to B and C.

BD, DC shall be together less than BA, AC, but shall contain an $\angle BDC$ greater than $\angle BAC$.



Produce BD to meet AC in E.

Then BA, AE, sides of $\triangle BAE$, are together greater than BE (I. 20).

To each of these add EC.

$\therefore$ BA, AC are greater than BE, EC.

Again CE, ED, sides of $\triangle CED$, are greater than CD.

To each of these add DB.

$\therefore$ BE, EC are greater than CD, DB.

And $\because$ BA, AC are greater than BE, EC, they are greater than CD, DB.

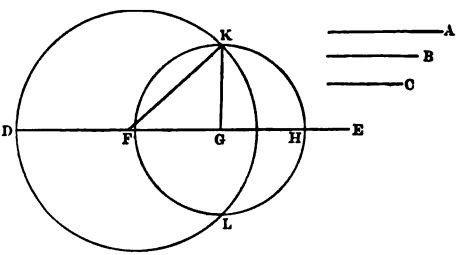
Next, $\because$ ext. $\angle BDC$ is greater than int. and opposite $\angle DEC$, and for the same reason $\angle DEC$ is greater than $\angle BAC$ (I. 16),

$\therefore$ $\angle BDC$ is greater than $\angle BAC$.

Wherefore if from the ends, &c.—Q.E.D.

Proposition XXII., Problem 8.

To make a triangle, of which the sides shall be equal to three given straight lines, any two of which are greater than the third.



Let A, B, C be the three straight lines, any two of which are greater than the third. It is required to make a Δ having its sides = A, B, C, each to each.

Take a straight line DE of unlimited length.

In DE make DF = A, FG = B, and GH = C (I. 3).

From centre F at distance FD describe circle DKL.

From centre G at distance GH describe circle HKL.

Join FK, GK.

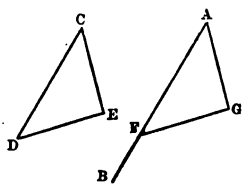
KFG is a triangle having its sides equal to A, B, C.

FK = FD = A
GK = GH = C
FG = B

∴ a Δ KFG has been described as required.—Q.E.F.

Proposition XXIII., Problem 9.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.



Let AB be the given straight line, A the point in it, and DCE the given rectilineal angle; it is required to make at point A a rectilineal angle equal to ∠ DCE. In CD, CE take any points D, E, and join DE.

Make the Δ AFG having its sides equal to the three straight

lines CD, CE, DE so that AF = CD, AG = CE, and FG = DE (I. 22).

The ∠ FAG shall be equal to the ∠ DCE.

∵ FA, AG, FG = DC, CE, DE, each to each.

∠ FAG = ∠ DCE (I. 8).

Wherefore at the given point A in AB the angle FAG has been made equal to the given angle DCE.—Q.E.F.

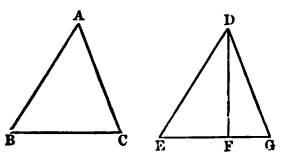
Proposition XXIV., Theorem 15.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle must be greater than the base of the other.

In the Δs ABC, DEF let AB = DE, and AC = DF, and let ∠ BAC be greater than ∠ EDF.

BC shall be greater than EF.

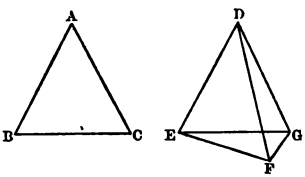
At the point D in DE make ∠ EDG = ∠ BAC (I. 23) and make DG = AC or DF, and join EG, GF.



Case I.—When EF falls on EG EF is a part of EG, ∴ EG is greater than EF.

Case II.—When EF falls below EG.

Then ∵ AB = DE, and AC = DG, and ∠ BAC = ∠ EDG



BC = EG (I. 4).

Again ∵ DG = DF

∴ ∠ DFG = ∠ DGF (I. 5)

∴ ∠ EFG is greater than ∠ DGF; much more then is ∠ EFG greater than ∠ EGF.

∴ EG is greater than EF (I. 19).

But EG = BC.

∴ BC is greater than EF.

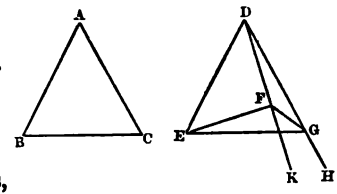
Case III.—When EF falls above EG.

Produce DF, DG to K and H.

$\angle$ GFK = FGH (I. 5).

Remainder of proof as in Case II.

Wherefore if two triangles, &c.—Q.E.D.

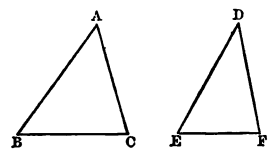


PROPOSITION XXV., THEOREM 16.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

In the $\triangle$s ABC, DEF, let AB = DE, and AC = DF, but let BC be greater than EF.

The $\angle$ BAC shall be greater than $\angle$ EDF.



For if not $\angle$ BAC must be either = $\angle$ EDF or less.

But $\angle$ BAC is not equal to $\angle$ EDF, for then the base BC would be equal to EF, but it is not (I. 4).

Neither is it less than $\angle$ EDF, for then the base BC would be less than EF, but it is not (I. 24).

$\therefore$ $\angle$ BAC is greater than $\angle$ EDF.

Wherefore if two triangles, &c.—Q.E.D.

PROPOSITION XXVI., THEOREM 17.

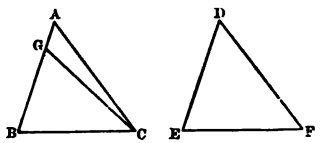
If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

In $\triangle$s ABC, DEF, let $\angle$ ABC = $\angle$ DEF and $\angle$ ACB = $\angle$ DFE.

I. Let the sides adjacent to the equal $\angle$ s in each be equal, *i.e.* let BC = EF.

Then shall AB = DE and AC = DF, and $\angle$ BAC = $\angle$ EDF.

For if AB be not equal to DE, one of them must be



greater than the other. Let AB be the greater and make BG = DE. Join GC.

Then in Δs GBC, DEF, GB = DE, BC = EF, and $\angle$ GBC = $\angle$ DEF.

$\therefore$ Δ GBC = Δ DEF and $\angle$ GCB = $\angle$ DFE (I. 4).

But $\angle$ ACB = $\angle$ DFE (hyp.).

$\therefore$ $\angle$ GCB = $\angle$ ACB, the less = the greater, which is impossible. $\therefore$ AB is not unequal to DE, *i.e.* it is equal to it.

Then in Δs ABC, DEF,

$\because$ AB = DE, and BC = EF, and $\angle$ ABC = $\angle$ DEF,

$\therefore$ AC = DF, and $\angle$ BAC = $\angle$ EDF (I. 4).

Next let the sides which are opposite equal angles in each triangle be equal: AB = DE.

Then shall AC = DF, and BC = EF, and $\angle$ BAC = $\angle$ EDF.

If BC be not equal to EF, let BC be the greater, and make BH = EF, and join AH.

Then in Δs ABH, DEF,

AB = DE, and BH = EF, and $\angle$ ABH = $\angle$ DEF.

$\therefore$ Δ ABH = Δ DEF, and $\angle$ AHB = DFE (I. 4).

But $\angle$ ACB = $\angle$ DFE (hyp.), $\therefore$ $\angle$ AHB = $\angle$ ACB, *i.e.* the exterior $\angle$ of Δ AHC = int. and opposite $\angle$ ACB, which is impossible (I. 16). $\therefore$ BC is not unequal to EF, *i.e.* it is equal to it.

Then in Δs ABC, DEF,

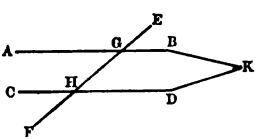
AB = DE, and BC = EF, and $\angle$ ABC = $\angle$ DEF

$\therefore$ AC = DF, and $\angle$ BAC = $\angle$ EDF (I. 4).

Wherefore if two triangles, &c.—Q.E.D.

Proposition XXVII., Theorem 18.

If a straight line falling on two other straight lines make the alternate angles equal to one another, the two straight lines shall be parallel to one another.



Let the straight line EF, falling on AB, CD make the alternate $\angle$s AGH, GHD equal, AB shall be parallel to CD.

For if not, AB and CD being produced will meet either towards B, D or A, C. Let them be produced and meet towards B, D in K.

Then GHK is a Δ, and its exterior $\angle$ AGH is greater than the interior and opposite $\angle$ GHD (I. 16).

But $\angle$ AGH also = $\angle$ GHD (hyp.), which is impossible.

$\therefore$ AB, CD do not meet when produced towards B, D.

In the same manner it may be shown that they do not meet towards A, C.

$\therefore$ AB and CD are parallel (def. 35).

Wherefore if a straight line, &c.—Q.E.D.

I. Let the sides adjacent to the e
i.e. let BC = EF.
Then shall AB = DE
∠ EDF.
For if AB be not e

greater than t
BG = DE.
Then in
∠ GBC =
∴ △ G
But
∴ ∠
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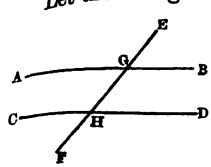
to it.
T

straight line EF, falling upon AB, *(1) the* exterior ∠ EGB = the *opposite* ∠ GHD, or (2) the *interior* ∠s on the same side, BGH, GHD *together* = two right angles:
shall be parallel to CD.
∠EGB = ∠ GHD (hyp.)
∠EGB also = ∠ AGH (I. 15).
∴ ∠ AGH = ∠ GHD; and they are alternate angles.
∴ AB is ∥ to CD.
II. ∵ ∠s BGH, GHD together = 2 rt. ∠s (hyp.)
∠s AGH, BGH „ „ „ (I. 13)
∴ ∠s AGH, BGH „ „ ∠s BGH, GHD.
Take away the common ∠ BGH, the remaining ∠ AGH = remaining angle GHD, and they are alternate angles.
∴ AB is ∥ to CD.
Wherefore if a straight line.—Q.E.D.

PROPOSITION XXIX., THEOREM 20.

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the straight line EF fall on the parallel straight lines AB, CD.



I. It makes the alternate angles AGH, GHD equal.

For if ∠ AGH be not equal to ∠ GHD, one of them must be greater than the other; let ∠ AGH be the greater.

To each of these angles add ∠ BGH.

Then the ∠s AGH, BGH are together greater than ∠s GHD.

But the ∠s AGH, BGH are together equal to two ∠s.

∴ the ∠s BGH, GHD are together less than two right ∠s (I. 13).

But if a straight line meet two straight lines, so as to make the two interior ∠s on the same side of it, taken together, less than two rt. ∠s, these straight lines being produced shall at length meet on that side on which are the angles which are less than two right angles.

∴ the straight lines AB, CD if produced far enough will meet.

But they never meet, since they are parallel by hypothesis.

∴ the ∠ AGH is not unequal to ∠ GHD; *i.e.* it is equal to it.

II. The exterior angle shall be equal to the interior and opposite angle on the same side.

Now the ∠ AGH = ∠ EGB (I. 15).

∴ the ∠ EGB = ∠ GHD (ax. 1).

III. And also the two interior angles on the same side equal to two right angles.

To each of the ∠s EGB, GHD add ∠ BGH.

Then the ∠s EGB, BGH = the ∠s GHD, BGH.

But the ∠s EGB, BGH = two rt. ∠s (I. 13).

∴ the ∠s GHD, BGH = two rt. ∠s.

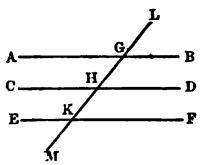
Wherefore if a straight line, &c.—Q.E.D.

Proposition XXX., Theorem 21.

Straight lines which are parallel to the same straight line are parallel to each other.

Let AB, CD be each of them ∥ to EF; AB shall be ∥ to CD.

Draw the straight line LM cutting AB, CD, EF in points G, H, K. Then ∵ LM cuts the ∥ lines AB, EF,



$\angle$ AGH = $\angle$ HKF.

And $\because$ LM cuts the $\parallel$ lines CD, EF,

Ext. $\angle$ GHD = int. and opposite $\angle$ HKF.

$\therefore$ $\angle$ AGH = $\angle$ HKF.

And these are alternate $\angle$s.

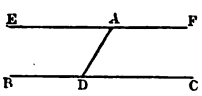
$\therefore$ AB is $\parallel$ to CD.

Wherefore straight lines, &c.—Q.E.D.

Proposition XXXI., Problem 10.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw through A a straight line $\parallel$ to BC.



In BC take any point D; join AD.

At point A in AD make $\angle$ DAE = $\angle$ ADC (I. 23).

Produce EA to F.

EF shall be $\parallel$ to BC.

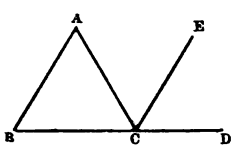
For $\because$ AD, meeting EF and BC, makes the alternate $\angle$s EAD, ADC equal; $\therefore$ EF is $\parallel$ to BC (I. 27).

$\therefore$ a straight line has been drawn through A $\parallel$ to BC.—Q.E.F.

Proposition XXXII., Theorem 22.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a $\triangle$, and let one of its sides BC be produced to D.



(1) The exterior $\angle$ ACD shall be equal to the two interior and opposite $\angle$s CAB, ABC.

(2) The three interior angles ABC, BAC, ACB shall = two rt. $\angle$s.

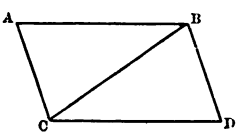
Through the point C draw CE $\parallel$ to AB (I. 31).

(1) Then ∵ AB is ‖ to CE, and AC meets them,
the alternate ∠s BAC, ACE are equal.
Again ∵ AB is ‖ to CE, and BD meets them,
the exterior ∠ ECD = the interior and opposite ∠ ABC (I. 29).
∴ ∠s ECD, ACE, *i.e.* exterior ∠ ACD = interior and opposite ∠s BAC, ABC.
(2) To each of these equals add ∠ ACB.
Then the ∠s ABC, BAC, ACB together = ∠ ACD, ACB = two rt. angles (I. 13).
Wherefore, if a side of any triangle, &c.—Q.E.D.

Proposition XXXIII., Theorem 23.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

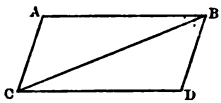
Let AB and CD be equal and ‖ straight lines joined towards the same parts by the straight lines AC and BD.



AC and BD shall be equal and parallel.
Join BC.
Then ∵ AB is ‖ to CD, and BC meets them,
∠ ABC = alternate ∠ DCB (I. 29).
And in the △s ABC, BCD,
AB = CD,
BC is common,
∠ ABC = ∠ BCD,
∴ Base AC = base BD, and
∠ ACB = ∠ CBD (I. 4).
And because the straight line BC meets the two straight lines AC, BD, and makes the alternate ∠s ACB, CBD equal,
AC is ‖ to BD (I. 27),
And it was shown to be equal to it.
Wherefore the straight lines, &c.—Q.E.D.

Proposition XXXIV., Theorem 24.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it.



Let ACDB be a ‖gram, of which BC is a diameter.

Then shall AB = DC, and AC = DB.

And ∠ BAC = ∠ CDB, and ∠ ABD = ∠ ACD,

And △ ABC = △ DCB.

Because AB is ‖ to CD, and BC meets them,

The alternate ∠s ABC, DCB are equal (I. 29).

And because AC is ‖ to BD, and BC meets them,

The alternate ∠s ACB, DBC are equal.

Then, in △s ABC, DCB,

∠ ABC = ∠ DCB, and ∠ ACB = ∠ DBC, and BC, a side adjacent to the equal angles in each triangle, is common.

∴ AB = DC, and AC = DB, and ∠ BAC = ∠ CDB (I. 26).

And △ ABC = △ DCB (I. 4).

Again, ∠ ABC = ∠ DCB, and ∠ DBC = ∠ ACB.

∴ whole ∠ ABD = whole angle ACD (ax. 2).

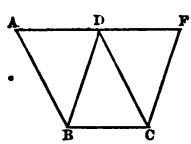
Wherefore the opposite sides and angles of a parallelogram are equal, and the diameter bisects it.—Q.E.D.

Proposition XXXV. Theorem.

Parallelograms on the same base and between the same parallels are equal to one another.

Let the ‖grams ABCD, EBCF be on the same base BC, and between the same ‖s AF, BC.

The ‖gram ABCD shall be equal to the ‖gram EBCF.

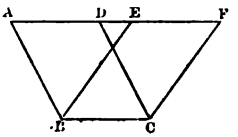


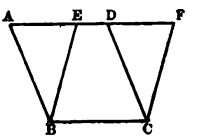
Case I. If the sides opposite to BC be terminated in the same point D,

Each of the ‖grams is double of △ DBC.

Therefore, they are equal to one another (ax. 6).

Cases II. and III. If the sides opposite the base be not terminated in the same point.





Because ABCD is a ||gram,

AD = BC.

Because EFBC is a ||gram,

EF = BC.

∴ AD = EF.

∴ The whole or the remainder AE = the whole or the remainder EF.

And AB = DC (I. 34).

And the exterior ∠ FDC = the interior and opposite ∠ EAB.

∴ Δ EAB = Δ FDC.

Take away the triangle FDC from the trapezium ABCF, and from the same figure take the Δ EAB, and the remainders, the ||grams ABCD, EBCF, are equal.

Wherefore, parallelograms on the same base, &c.—Q.E.D.

Proposition XXXVI. Theorem.

Parallelograms on equal bases and between the same parallels are equal to one another.

Let ABCD, EFGH be ||grams on equal bases BC, FG, and between the same ||s AH, BG.

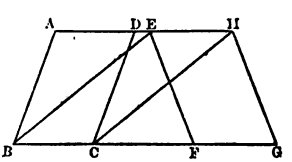
The ||gram ABCD shall be equal to the ||gram EFGH.

Join BE, CH.

Then ∵ BC = FG (hyp.)

and FG = EH

∴ BC = EH



and they are parallels, and joined towards the same parts by the straight lines BE, CH.

But straight lines which join the extremities of equal and ‖ straight lines are themselves equal and ‖, ∴ BE, CH are both equal and ‖ (I. 33).

∴ EBCH is a ‖gram.

And it is equal to ABCD, ∵ they are on the same base BC and between the same ‖s BC, AH (I. 35).

For the same reason it is equal to ‖gram EFGH.

∴ ‖gram ABCD = ‖gram EFGH (ax. 1).

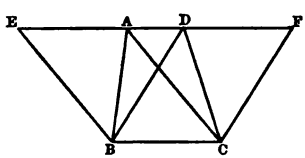
Wherefore parallelograms, &c.—Q.E.D.

Proposition XXXVII. Theorem.

Triangles upon the same base and between the same parallels are equal to one another.

Let the △s ABC, DBC be on the same base BC and between the same ‖s AD, BC.

△ ABC shall be equal to △ DBC.



Produce AD both ways.

Through B draw BE ‖ to CA to meet AD in E.

Through C draw CF ‖ to BD to meet AD in F (I. 31).

Then EBCA and FCBD are ‖grams (def.) and ‖gram EBCA = ‖gram FCBD,

∵ they are on the same base BC and between the same ‖s BC, EF (I. 35).

And △ ABC is half of ‖gram EBCA, because the diameter AB bisects the ‖gram (I. 34).

For the same reason △ DBC is half of ‖gram FCBD.

But the halves of equal things are equal (ax. 7).

Wherefore triangles, &c.—Q.E.D.

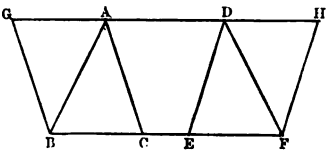
Proposition XXXVIII. Theorem.

Triangles on equal bases and between the same parallels are equal to one another.

Let the △s ABC, DEF be on equal bases BC, EF, and between the same ‖s BF, AD.

△ ABC shall be equal to △ DEF.

Produce AD both ways.



Through B draw BG ∥ to CA to meet AD in G.

Through F draw FH ∥ to ED to meet AD in H (I. 31).

Then GBCA and DEFH are ∥^{grams} (def.).

And they are equal, ∵ they are on equal bases BC, EF, and between the same ∥s BF, GH (I. 36).

And △ ABC is half of ∥^{gram} GBCA.

And △ DEF is half of ∥^{gram} DEFH (I. 34).

But the halves of equal things are equal (ax. 7).

∴ △ ABC = △ DEF.

Wherefore triangles, &c.—Q.E.D.

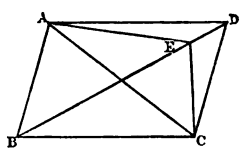
Proposition XXXIX. Theorem.

Equal triangles on the same base and on the same side of it a re between the same parallels.

Let the equal △s ABC, DBC be on the same base BC and on the same side of it.

They shall be between the same parallels.

Join AD.



AD shall be ∥ to CD.

If not, through A draw AE ∥ to BC, and meeting BD at E (I. 31),

Join EC.

Then △ ABC = △ EBC,

∵ they are on the same base BC and between the same ∥s BC, AE (I. 37).

But △ ABC = △ DBC (hyp.).

∴ △ EBC = △ DBC.

The less to the greater, which is impossible.

∴ AE is not ∥ to BC.

In the same manner it can be shown that no other line through A but AD is ∥ to BC.

∴ AD is ∥ to BC.

Wherefore equal triangles, &c.—Q.E.D.

But straight lines which join the
‖ straight lines are themselves e
both equal and ‖ (I. 33).

∴ EBCH is a ‖gram.

And it is equal to ABC
BC and between the sam

For the same reason

∴ ‖gram ABCD =

Wherefore parall

straight line and
same parallels.

equal bases BC, EF, in
the same straight line BF,
and on the same side of
it; they shall be between
the same parallels.

Join AD.

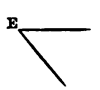
AD shall be ‖ to BF.

PROP

Triangles upo

through A draw AG ‖ to BF, meeting ED at G

Let the
between th



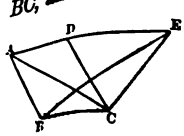
Then △ ABC = △ GEF (I. 38).
but △ ABC = △ DEF (hyp.).
∴ △ GEF = △ DEF (ax. 1),
the less to the greater, which is impossible.
∴ AG is not ‖ to BF.
In the same manner it can be shown that no other straight
line through A but AD is ‖ to BF.
∴ AD is ‖ to BF.
Wherefore equal triangles, &c.—Q.E.D.

‖gr

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels the parallelogram shall be double of the triangle.

Let the ‖gram ABCD and the △ EBC be on the same base
BC, and between the same ‖s BC, AE.



The ‖gram ABCD shall be double of △ EBC.

Join AC.

Then △ ABC = △ EBC, ∵ they are on the same base and between the same ‖s (I. 37).

nd ‖gram ABCD is double of Δ ABC, ∵ diameter AC the ‖gram (I. 34).

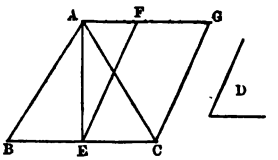
n ABCD is also double of Δ EBC.

ore if a ‖gram, &c.—Q.E.D.

Proposition XLII., Problem 11.

escribe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given Δ, and D the given ∠ : it is required to describe a ‖gram equal to Δ ABC, and having one of its ∠s equal to ∠ D.



Bisect BC at E (I. 10).

Join AE.

At the point E in the straight line EC, make ∠ CEF = ∠ D (I. 23).

Through A draw AFG ‖ to BC.

Through C draw CG ‖ to EF.

FECG is a ‖gram by construction,

And ∵ BE = EC (cons.),

Δ ABE = Δ AEC ∵ they are on equal bases, and between the same ‖s BC, AG (I. 38).

∴ Δ ABC is double of Δ AEC.

But ‖gram FECG is also double of Δ AEC (I. 41).

∵ they are on the same base EC, and between the same ‖s EC, AG.

∴ ‖gram FECG = Δ ABC (ax. 6), and it has one of its ∠s CEF equal to given ∠ D.

Wherefore a ‖gram has been described, &c.—Q.E.D.

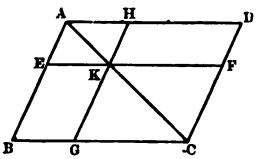
Proposition XLIII., Theorem 32.

The complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.

Let ABCD be a ‖gram, of which AC is the diameter, and EH, GF ‖grams about AC; and BK, KD the other ‖grams

which make up the whole figure ABCD, and are therefore called complements.

The complement BK shall be equal to the complement KD.



Because ABCD is a ‖gram, and AC its diameter, Δ ABC = Δ ADC.

Again, ∵ EH is a ‖gram, and AK its diameter, Δ AEK = Δ AHK (I. 34).

For the same reason Δ KGC = Δ KFC.

∴ Δs AEK, KGC are together equal to Δs AHK, KFC.

But Δ ABC was shown to be equal to Δ ADC.

∴ the remainder, the complement BK = the remainder the complement KD.

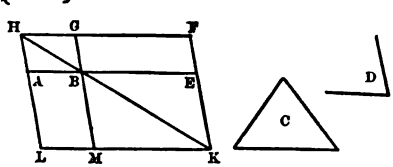
Wherefore the complements, &c.—Q.E.D.

Proposition XLIV., Problem 12.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given Δ, and D the given ∠: it is required to apply to AB a ‖gram equal to Δ C, and having an angle equal to ∠ D.

Make a ‖gram BEFG = Δ C, and having ∠ EBG = D, so that its side BE may be in the same straight line with AB (I. 42).



Produce FG to H.

Through A draw AH ‖ to BG or EF (I. 31).

Join HB.

Then ∵ HF meets the ‖s AH, EF, the ∠s AHF, HFE are together equal to two rt. ∠s (I. 29).

∴ the ∠s BHF, HFE are together less than two rt. ∠s.

∴ HB and FE will meet if produced towards B and E (post. 6).

Let them meet at K.

Through K draw KL ‖ to EA or FH (I. 31).

Produce HA, GB to the points L, M.

Then HLKF is a ‖gram. and HK its diameter, and AG, ME are ‖grams about HK; and LB, BF are complements.

∴ LB = BF.

But BF = △ C.

∴ LB = △ C.

And ∵ ∠ GBE = ∠ ABM (I. 15), and also

∠ GBE = ∠ D;

∴ ∠ ABM = ∠ D.

Wherefore to the given straight line AB the ‖gram LB is applied, equal to △ C, and having ∠ ABM = ∠ D.—Q.E.F.

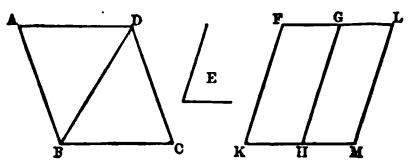
Proposition XLV., Problem 13.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given figure, and E the given ∠ : it is required to describe a ‖gram = ABCD, and having an angle = ∠ E.

Join DB.

Describe ‖gram FH = △ ADB, and having ∠ FKH = ∠ E (I. 42).



To the straight line GH apply the ‖gram GM = △ DBC, and having ∠ GHM = ∠ E (I. 44).

The figure FKML shall be the required ‖gram.

∵ ∠ E = each of ∠s FKH, GHM (cons.)

∠ FKH = ∠ GHM (ax. 1).

To each of these equals add ∠ KHG;

∴ ∠s FKH, KHG = ∠s KHG, GHM (ax. 2).

But ∠s FKH, KHG = two rt. ∠s (I. 29);

∴ ∠s KHG, GHM = two rt. ∠s,

∴ KHM is a straight line (I. 14).

And ∵ HG meets the ‖s KM, FG,

The ∠ MHG = alternate ∠ HGF (I. 29).

To each of these add ∠ HGL.

∴ ∠s MHG, HGL = ∠s HGF, HGL.

But MHG, HGL = two right ∠s (I. 29);

∴ ∠s HGF, HGL = two right ∠s,

∴ FG is in the same straight line with GL (I. 14).

And because KF is ‖ to HG, and HG to ML (cons.),

KF is ‖ to ML (I. 30).

And KM, FL are ‖s (cons.).

∴ KFLM is a ‖gram.

And ∵ Δ ABD = ‖gram HF, and Δ DBC = ‖gram GM; the whole figure ABCD = whole ‖gram KFLM.

Wherefore the ‖gram, &c.—Q.E.F.

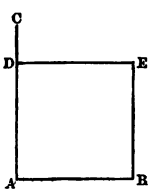
Proposition XLVI., Problem 14.

To describe a square on a given straight line.

Let AB be the given straight line:

It is required to describe a square on AB.

From point A draw AC at rt. angles to AB (I. 11).



Make AD = AB (I. 3).

Through D draw DE ‖ to AB.

Through B draw BE ‖ to AD.

ADEB shall be a square,

For it is by construction a ‖gram.

∴ AB = ED, and AD = BE (I. 34).

But AB = AD.

∴ AB, BE, ED, DA are all equal.

∴ ADEB is equilateral.

Again ∵ the straight line AD meets the ‖s AB, DE.

∠s BAD, ADE together = two rt. angles.

But $\angle$ BAD is a rt. $\angle$,
$\therefore$ $\angle$ ADE is a rt. $\angle$.
But the opposite $\angle$s of a $\|^{gram}$ are equal.
Therefore, each of the $\angle$s ABE, BED is a rt. $\angle$.
$\therefore$ ADEB is rectangular, and also equilateral.
$\therefore$ it is a square, and it is described on AB.—Q.E.F.

Proposition XLVII., Theorem 33.

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled triangle, having BAC a rt. $\angle$.

The square described on BC shall be equal to sqq. on BA, AC.

On BC describe the square BDEC.

On BA, AC describe the sqq. GB, HC (I. 46).

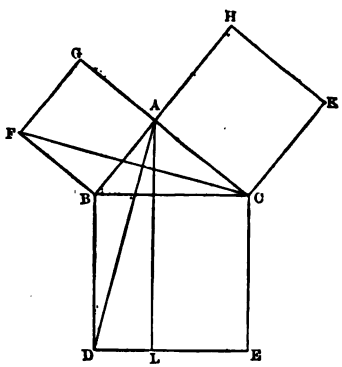
Through A draw AL $\|$ to BD or CE (I. 31).

Join AD, FC.

Then $\because$ BAC and BAG are both rt. $\angle$s,

CA is in the same straight line with AG (I. 14).

For the same reason AB and AH are in the same straight line.



Now $\angle$ DBC = $\angle$ FBA, for each of them is a rt. $\angle$.

Add to each $\angle$ ABC.

$\therefore$ whole $\angle$ DBA = whole $\angle$ FBC (ax. 2).

And $\because$ AB = FB, BD = BC, and $\angle$ DBA = $\angle$ FBC,

$\therefore$ Δ ABD = Δ FBC (I. 4).

Now the $\|^{gram}$ BL is double of Δ ABD, because they are on the same base BD and between the same $\|$s (I. 41).

And the square GB is double of Δ FBC, $\because$ they are on the same base FB, and between the same $\|$s (I. 41).

But the doubles of equals are equal to one another (ax. 6).

$\therefore$ $\|^{gram}$ BL = square GB.

In the same manner, by joining AE, BK, it can be shown that the $\|^{gram}$ CL = sq. CH.

$\therefore$ the whole square BDEC = two sqq. GB, HC.

$\therefore$ the sq. described on BC = sqq. on BA, AC.

Wherefore in any right-angled triangle, &c.—Q.E.D.

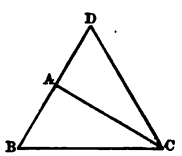
Proposition XLVIII., Theorem 34.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, a side of Δ ABC, be equal to the squares described on the other sides BA, AC.

$\angle$ BAC shall be a right angle.

From point A draw AD at right angles to AC (I. 11).



Make AD = BA (I. 3).

Join DC.

Then $\because$ DA = BA,

sq. on DA = sq. on BA.

To each of these add sq. on AC.

$\therefore$ The sqq. on DA, AC = sqq. on BA, AC (ax. 2).

And $\because$ $\angle$ DAC is a right $\angle$.

The square on DC = sqq. on DA, AC (I. 47), and by hypothesis

sq. on BC = sqq. on AB, AC.

$\therefore$ sq. on DC = sq. on BC (ax. 1).

$\therefore$ side DC = side BC.

$\therefore$ in Δs BAC, DAC,

$\because$ DA = AB (cons.)

AC is common,

and DC = BC,

$\therefore$ $\angle$ DAC = $\angle$ BAC (I. 8).

But DAC is a right angle.

$\therefore$ BAC is a right angle.

Wherefore if a square, &c.—Q.E.D.

LECTURE XVIII.

BOOK II.

PROPOSITIONS 1–4.

In the second Book, Euclid, continuing the subject of equivalence of area, investigates the relations between the rectangles contained by straight lines divided in various ways.

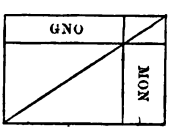
An oblong, or rectangle, we know is 'a figure which has all its angles right angles, but not all its sides equal' (I. def. 31), and further, since it is one of those figures classed together as parallelograms, 'its opposite sides are equal' (I. 34). The book opens with two definitions:—

Def. I.—'Every rectangle is said to be contained by any two of the straight lines which contain one of the right angles.'

For since the opposite sides are equal, and the angles are all right angles, these two lines being given, its area is determined, and the rectangle can be constructed. With this fact you are doubtless quite familiar from arithmetic, since 'any two of the straight lines which contain one of the right angles' of a rectangle will give its *length* and *breadth*, which, as you know, are the measurements required in order to find the area.

Def. II.—'In every parallelogram any of the parallelograms about the diameter, together with the two complements, is called a gnomon.'

Gnomon is a Greek word signifying a carpenter's rule. The figure is probably so called from its resemblance to a rule bent at an angle.

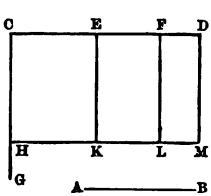


The first three propositions develop one theorem, of which Prop. I. gives the general enunciation, and Props. II. and III. two important cases, to which Euclid has thought it necessary to call attention.

Prop. I.—'If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.'

Let AB and CD be two straight lines, one of which, CD, is divided at points E and F: the rectangle contained by AB and CD shall be equal in area, or equivalent to, the sum of the areas of the three rectangles contained by a line equal to AB and each of the parts into which CD is divided.

Assuming this theorem to be true, we must be able to divide the rectangle contained by AB, CD into smaller rectangles, which will be equal to those contained by AB and the several parts of CD, and these rectangles *ought* to occupy the entire area of the rectangle contained by AB, CD. Let us see if this is the case. The rectangle contained by AB, CD must first be constructed. To begin with, we require a right angle. By I. 11 we can draw a straight line at right angles to CD, then, cutting off a part CH (I. 3) equal to AB, we obtain two sides of the rectangle required. To complete it lines parallel to CH, CD must be drawn through D and H, these will meet at a point M. Then CM is by construction a parallelogram; also it has a right angle HCD. From the corollary to I. 46 we learn that a parallelogram having one right angle has *all* its angles right angles, therefore CM is a rectangle, and it is contained by CD and by CH equal to AB. Can CM be divided into rectangles equal to those contained by AB and the several parts of CD?



Since CH is equal to AB, and is at right angles to CE, by drawing EK parallel to CH to meet HM in K, a rectangle CK is constructed equal to the rectangle contained by AB and CE.

Again, since EK and DM are each equal to CH or AB (I. 34), by drawing FL parallel to CH the remaining portion EM will be divided into two rectangles equal to those contained by AB, EF, and AB, FD.

Thus the whole rectangle CM, that is the rectangle contained by AB, CD = the sum of the rectangles CK, EL, FM, which were constructed severally equal to the rectangles contained by AB and the several parts of CD.—Q.E.D.

Summary of Proposition I., Theorem 1.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

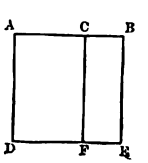
Rect. AB, CD = sum of rects. AB, CE, AB, EF, AB, FD.

Cons.—Describe rect. contained by the whole lines (I. 11, I. 3).

Divide it into rects. contained by whole and each of the parts (I. 31).

Proof.—These together make up and are therefore equal to rect. contained by the whole lines.

The second proposition is thus enunciated: 'If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.'



This is, as has been said, merely a particular case of the last proposition, for, supposing the given lines to be equal, then the 'square on the whole line' is equal to 'the rectangle contained by the two given lines,' and 'the rectangles contained by the whole and each of the parts' are the same as the 'rectangles contained by the undivided line and the several parts of the divided line.' The

demonstration is, as we should expect, similar to that of Prop. I. The construction is rather simpler, for, taking AB as the given line, and C the point of section, by I. 46 we can describe a square on AB, which is divided into the required rectangles by drawing a line parallel to its sides through point C.

Summary of Proposition II., Theorem 2.

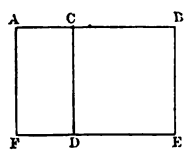
If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

(Sq. on AB = sum of rect. AB, BC and rect. BA, AC.)

Cons.—Describe square on whole line (I. 46). Divide it into rects. contained by whole and each of the parts (I. 31).

Proof.—These rects. will make up and are therefore equal to whole square.—Q.E.D.

Proposition III. shows that a straight line AB, being divided at any point C, the rectangle contained by the whole line and one of the parts (rect. AB, BC) shall be equal to the square on that part (sq. on BC), together with the rectangle contained by the two parts (rect. AC, CB).



To adapt this enunciation to the conditions of II. 1 we have only to suppose that AB is divided so that BC is equal to the second or 'undivided line,' then the 'rectangle contained by the whole and one of the parts = the 'rectangle contained by two given lines,' and the square on BC = the rectangle contained by one of the parts and the undivided line, &c.

The demonstration is the same as in II. 1 and 2, making the obvious difference in the construction in accordance with the terms of the enunciation.

SUMMARY OF PROPOSITION III., THEOREM 3.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

(Rect. AB, BC = sq. on BC and rect. AC, CB.)

Cons.—Describe square on BC (I. 46).

Complete rect. AB, BC (I. 31).

Proof.—Small rect. = rect. AC, CB ∵ AF = BC (*cons.*)

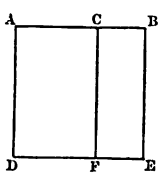
Figure BCDE = sq. on BC (*cons.*)

Small rect. and the square together make up and = large rectangle AB, BC.

Proposition IV. shows that 'if a straight line be divided into two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts;' or, supposing the data to be two straight lines, the enunciation may be stated: 'The square on the sum of two straight lines = the sum of the squares on those lines, together with twice the rectangle contained by them.'

Sq. on AB = sqq. on AC and CB and 2 rect. AC, CB.

On AB describe the square ABED (I. 46).



Can we divide this square into parts equal to the squares and rectangles required? First, it is evident that by drawing a line CF parallel to AD or BE through the point C the square is divided into two rectangles, each of which must be larger than the squares on the parts, since AD and BE are each equal to AB the whole line. Examine one of these rectangles AF, and you will see that it is contained by the whole line AB (AD = AB) and AC one of the parts, and therefore by the last proposition it is equal to the square on AC, together with the rectangle AC, CB.

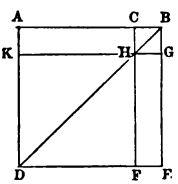
Similarly the rectangle CE is contained by AB, BC (BE

= AB), and is therefore equal to the square on BC, together with the rectangle AC, CB (II. 3).

That is, the two rectangles into which the square on AB is divided are together equal to the squares on AC, CB, and twice the rectangle AC, CB.—Q.E.D.

Again, it is easy to see that the rectangles AF, CE, into which the square on AB is divided, may be subdivided so as to form the four parts mentioned in the enunciation.

For taking BG, a part of BE equal to BC, and drawing GHK parallel to one of the opposite sides of the square, we obtain:—



CG = square on BC, ∵ BC = BG and ∠ CBG is a rt. ∠

KF = square on AC, ∵ AD = DE (I. 46), and AK = BG = BC = FE (I. 34); and ∴ KD and DF, the remaining parts, are equal; ∠ KDF is a rt. ∠.

AH and HE, each contained by lines equal to AC, CB, and therefore together equal to twice the rectangle AC, CB; these four together make up AE, that is the square on AB.

Also, if we draw BD the diameter of AE, you will see that it passes through the point H and forms the diameters of KF and CG; hence these squares are parallelograms about the diameter of AB, and AH and FE are the complements. This fact probably suggested Euclid's construction, which differs slightly from the one we have just considered. Having drawn the square on AE and the line CF, he draws the diameter BD, and then through H, the point at which BD cuts CF, draws GHK parallel to AB or DE. The figure resulting is of course the same, but the proof differs considerably; the reasoning by which Euclid shows that CG is a square is very ingenious. (See summary, Euclid's method, below.)

A particular case of this proposition is important. 'If a straight line be bisected, the square on the whole line is equal to four times the square on half the line.'

Since the line is divided into two equal parts, the rectangle contained by the two parts will be the same as the square on half the line, hence the 'squares on the two parts' = twice the square on half the line, and twice the rectangle contained by the parts = twice the square on half the line; that is, the square on the whole line = four times the square on half the line.

Summary of Proposition IV., Theorem 4.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

(Sq. on AB = sqq. on AC, CB, and twice rect. AC, CB.)

Cons.—Draw a line through the pt. of section parallel to the sides of the square on the whole line.

Proof.—Each of the rects. into which the sq. on AB is divided = the square on one of the parts and rect. AC, CB (II. 3).

∴ the 2 rects. together, that is the square on AB, = the squares on the two parts and twice rect. AC, CB.

2nd Method.

Cons.—From BE cut off BG = BC.

Through pts. C and G draw lines ‖ to opposite sides of sq. on AB.

Proof.—AE, that is sq. on AB, = sum of CG and KF (= sqq. on AC, CB), and rects. AH, HE (= twice rect. AC, CB).

3rd (Euclid's) Method.

Cons.—Draw CF ‖ AD, join BD.

Through pt. H, at which these lines intersect, draw GHK ‖ to AB.

Proof.—Ext. ∠ CHB = int. and opp. ∠ ADB (I. 29).

= ∠ ABD., *i.e.* ∠ CBH (I. 5, ax. 4).

∴ BC = CH (I. 6).

$\therefore$ since CG is a rect. parallelogram by cons. it is a square (I. 34).

For the same reason KF is a sq. on KH = AC (I. 34).

Also AH, HE being complements are equal (I. 43).

And AH = rect. AC, CB $\because$ CH = BC.

$\therefore$ four figures KF, CG, AH, and HE = squares on AC, CB and twice rect. AC, CB, and together make up sq. on AB. —Q.E.D.

LECTURE XIX.

PROPOSITIONS 5, 6, AND 8.

WE have seen that the first three propositions may be included under one enunciation. Props. 5, 6, and 8 form a second group of three propositions, each of which is a particular case of the general law, that 'the rectangle contained by two straight lines, together with the square on half their difference, is equal to the square on half their sum.'

To make this clear let us take a concrete example:—

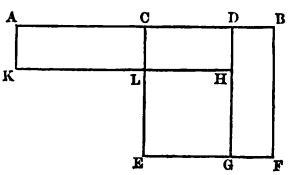
Given two straight lines, 6 in. and 2 in. long.

Area of the rectangle contained by the two straight lines $= 6 \times 2 = 12$ sq. in.

Area of square on half their difference $= (\frac{6-2}{2})^2 = 2^2 = 4$ sq. in.

i.e. area of rectangle, + square on half the difference $= 16$ sq. in.; area of square on half their sum $= (\frac{6+2}{2}) = 4^2 = 16$ sq. in.

We proceed to the geometrical proof of this law. Let AD and DB be the two given straight lines, and let DB be in the same straight line with AD.



Then the line AB is the sum of AD and DB, bisect it in C, and on CB describe a square CBFE (I. 46). This is the square on half of their sum. To describe the rectangle ADDB, draw DG parallel to CE or BF, one of the opposite sides of the square, and cut off from it DH equal to DB. AD and DH will be the sides containing a right angle; complete the rectangle as in former propositions.

It remains to construct the square on half the difference of the two straight lines.

Now AD = the difference + DB.

AB = AD + DB = the difference + 2 DB.

Therefore half AB, that is CB = half the difference and DB, and CB = CD + DB.

Therefore CD = half of the difference of the two lines.

From the construction in the last proposition (II. 4), we know that LG is a square, and its side LH = CD.

Then LG is the square on half the difference of AD and DB; and CF, the square on half the sum, is equal to LG, the square of half the difference, together with the rectangle CH, a part of rectangle AD, DB, and the rectangle DF.

Is the rectangle DF equal to AL the remaining part of AH?

GF = DB, and BF = CB (I. 46) = AC.

Therefore the rectangle AL = DF.

Consequently CF, the square on half the line, = the rectangle AD, DB (AH), together with the square on CD (LG).

In constructing the figure for this demonstration, we took care to place AD and DB in one straight line, in order to obtain a measure of their sum; and the whole line AB was bisected at C, and divided into two unequal parts at D Hence, supposing one line AB to be given, the proposition may be enunciated:—

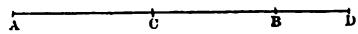
If a straight line (AB) be divided equally (at C) and unequally (at D), the rectangle contained by the unequal parts (ADDB), together with the square on the line between the points of section (CD), is equal to the square on half the line. This is the form of enunciation which Euclid adopts in the fifth proposition; his demonstration differs slightly from that given above, for having described the square on CB, he draws its diameter and adopts the construction followed in II. 4.

Proposition VI. presents the second case of the general enunciation given above.

If a straight line AB be bisected in C, and produced to any point D, the rectangle contained by the whole line thus produced, and the part of it produced (rect. AD, DB),

together with the square on half the line bisected (CB), is equal to the square on (CD) the straight line, which is made up of the half and the part produced.

That this is a particular case of the enunciation referred to will easily be seen.

Let AD and DB be the 

given lines; then AB is their difference. Since AB is bisected in C, CB is half their difference.

Sum of the two lines = the difference + 2 BD.

Therefore half their sum = half their difference + BD,

= CD.

The construction and proof are similar to that given, making the necessary change in constructing the rectangle ADDB.

Euclid adopts the same construction as before.

We pass to Proposition VIII., the third of the group. If a straight line AB be divided at C into any two parts, four times the rectangle contained by the whole line and one of the parts (4 AB, BC), together with the square on AC, the other part, is equal to the square on the straight line, which is made up of the whole and that part (AB produced to D, so that BD = BC).

Again, supposing AB, BC to be two given lines, we may state the enunciation thus:—

Four times the rectangle contained by the two lines, together with the square on their difference (AC), is equal to the square on their sum (AB + BC).

From the corollary to II. 4, we learn that the square on any line is equal to four times the square on half that line. Whence we obtain this equation:—

Four times the rectangle contained by the two lines, together with four times the square on half their difference = four times the square on half their sum.

Hence it is clear that this proposition also may be inferred from the enunciation which we have given of II. 5.

It may also easily be proved as an independent demonstration by either of the methods indicated in the last three propositions.

Summary of II. 5, 6, and 8.

General Enunciation.

The rectangle contained by two straight lines with the square on half their difference = the square on half their sum.

Rect. AD, DB and sq. on CD = sq. on CB.

Euclid's Enunciation II. 5.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Cons. (*Euclid's Method*).—On CB describe a sq. : draw its diagonal BE.

Draw DG ‖ to EC cutting BE.

Through H the point of section draw KLM ‖ to AB.

Through A draw AK ‖ to CL.

Proof.—Comp. CH = Comp. HF (I. 43).

$$AL = CM = DF,$$

$$\therefore AH(AL + CH) + LG = CH + DF + LG$$

$$= \text{sq. on CB.}$$

Cons. (*Another Method*).—Describe sq. on CB (half the line).

From DG drawn ‖ to CE cut off DH = DB,

Complete the rect. AD, DB.

Proof.—$AH(AL + CH) + LG = CH + DF + LG,$

$$= \text{sq. on CB.}$$

Euclid's Enunciation II. 6, Theorem 6.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line, which is made up of the half and the part produced.

Rect. AD, DB and sq. on CB = sq. on CD.

Cons. (*Euclid's Method*).—Describe sq. on CD,

Proceed as in II. 5.

Proof.—As in II. 5. (The part of rect. AD, DB outside of the square on CD = one of the complements.)

Another Method.—In constructing rect. AD, DB cut off DM = DB from DF a side of the square.

Proof.—As in II. 5. AL = CH = HF,

AM (AL + CM) + LG = HF + CM + LG
= sq. on CD.

EUCLID'S ENUNCIATION II. 8, THEOREM 8.

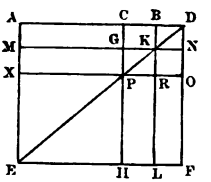
If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line, which is made up of the whole and the first part.

4 (rect. AB, BC) and sq. on AC = sq. on AD (AB + BC).

Cons. (*Euclid's Method*).—Produce AB to D making BD = BC,

On AD describe a square.

Complete the figure as in preceding prop. drawing two ‖s each way.



Proof.—CK = BN, GR = RN } I. 34 and I. 36
CK = RN I. 43

∴ The four small squares are equal.

For the same reasons the four small rects. are equal.

Rect. AB, BC = AK = 1 small rect. and 1 small square.

∴ 4 (rect. AB, BC) = 4 small rects. and 4 small squares.
= gnomon AOH.

Gnomon AOH and XH (*i.e.* sq. on AC) = sq. on AD.

∴ 4 (rect. AB, BC) and sq. on AC = sq. on AD.

Cons. (*Another Method.*)—Produce AB to D, making BD = BC,

On AD describe a sq.,

In DF cut off DN, NO each = BC.

Complete figure as in previous prop., drawing two ‖s each way.

Proof.—The four small rects. and the four small squares are equal (cons. and I. 34).

Rect. AB, BC, *i.e.* AK = 1 small rect. and 1 small sq.

∴ 4 (rect. AB, BC) = 4 small rects. and 4 small sqq.

∴ (4 rect. AB, BC) and square on AC, *i.e.* XH = 4 small sqq., and rects. together with XH = sq. on AD.

LECTURE XX.

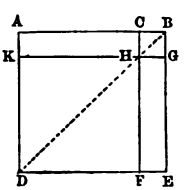
PROPOSITIONS 7, 9, AND 10.

PROPOSITION VII. is thus enunciated by Euclid: 'If a straight line (AB) be divided (at C) into any two parts, the squares on (AB and BC) the whole line, and one of the parts, are equal to twice the rectangle (AB, BC) contained by the whole and that part, together with the square on (AC) the other part.'

Construct a figure similar to those in the preceding propositions.

Then from the demonstration previously given we know that the rectangles AG, BF each = rect. AB, BC,

KF = square on AC,
CG = square on BC,
AE = square on AB.



You will notice that CG is common to both of the rectangles AG and BF. Therefore AG and BF together, that is twice the rectangle AB, BC, = the gnomon AGF together with the square on BC.

To each of these equals let us add KF (the square on AC), which completes the figure. Then we have this equation: Twice the rectangle AB, BC and the square on AC (KF) = the square on AB (gnomon and KF), and square on BC.—Q.E.D.

Now, instead of considering AC, CB as parts of a divided line let us consider that two straight lines are given, equal in length to AB and BC—then AC is clearly their difference, and the proposition may be enunciated thus:

The square on the difference of two straight lines (sq. on AC), together with twice the rectangle contained by these lines (2 AB, BC) = the sum of the squares on the two lines (squares on AB, BC).

From II. 4 (second enunciation) we learn that 'The square on the sum of two lines = the sum of the squares on those two lines together with twice the rectangle contained by them.' That is,

Sq. on difference and twice the rect. contained by two lines = sum of sqq. on two lines (II. 7).

Sq. on sum of two lines, less twice the rect. contained by two lines = sum of sqq. on two lines (II. 4).

Adding these two equations (ax. 2), we infer that,

Sq. on diff. and sq. on sum of two lines = twice the sum of the sqq. on two lines. This corollary will be found to include props. 9 and 10; enunciated by Euclid as follows (*a*. II. 9): If a straight line (AB) be divided into two equal and also into two unequal parts (at C and D), the squares on the two unequal parts (sqq. on AD, DB) are together double of the squares on half the line, and on the line between the point of section.

Sqq. on AD, DB = twice the sqq. on AC and CD.

Supposing AC and CD to be the two given lines, then since CB = AC (cons.), DB is their difference and AD their sum. Hence by the corollary given above we have:—

The squares on AD the sum, and DB the difference = twice the sum of the squares on the given lines AC, CD.

(*β*. II. 10.) 'If a straight line AB be bisected at C and produced to any point D, the square on the whole line thus produced (AD), and the square on the part of it produced (BD), are together double of the squares on half the line bisected (AC), and of the line made up of the half and the part produced (CD).'

Sqq. on AD, DB = twice the sqq. on AC, CD.

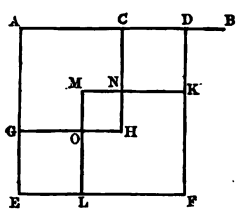
Let AC and CD be the two given lines, then AD is their sum, and BD their difference (CB = AC cons.). Hence by this corollary—

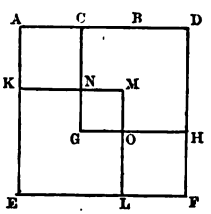
Sqq. on AD (sum), and DB (difference) = twice the sqq. on AC, CD (given lines).

These theorems may also be proved by the method adopted in the eight previous propositions, thus: Describe the square on the whole line AD. Construct at opposite angles of this square two squares equal to the sq. (1) on AC II. 9, (2) on CD II. 10.

These squares will overlap, forming a small square equal to the square on DB.

The two remaining squares completing the figure may easily be shown to be double of the square on AC (II. 9) or CD (II. 10).





Hence it is evident that the sq. on AD = twice the squares on AC and CD, less the square on DB, *i.e.* twice the sqq. on AC and CD = sqq. on AD and DB.—Q.E.D.

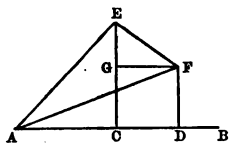
It remains to consider Euclid's demonstration of these propositions, which is very interesting and ingenious.

II. 9.

Sqq. on AD and DB = twice the squares on AC, CD.

We are required to show that one pair of squares are equal in area to two other pairs. Now each of these pairs may be represented by the square on a single line (I. 47) by drawing each pair of lines at right angles, and joining their extremities, thus forming a right-angled triangle, the square on whose hypothenuse will be equal to the squares on the lines containing the right angle. By this means the six squares in the enunciation will be represented by three.

Again, assuming the truth of the theorem, one of these squares is equal to the other two, consequently the lines on which they are to be constructed will form a right-angled triangle (I. 48). Let us see if this is the case.



From D draw a line at right angles to AB, and make DF = DB, join FA. Then sq. on FA = sqq. on ADDF, *i.e.* AD, DB. Similarly, drawing a line from C at right angles to AB make CE = AC and join AE.

Sq. on AE = sqq. on AC, CE, *i.e.* twice sq. on AC.

Also FA, AE form two sides of a triangle, of which we obtain the third by joining EF.

Is the sq. on EF = twice the sq. on CD?

Through F draw FG parallel to CD meeting CE in G. Then GD is a parallelogram by construction, and GF = CD (I. 34). Also, since CE = AC = CB and CG = DF or DB (I. 34), GE = CD, and the angle at G is a right angle (I. 29). Therefore the square on EF = the sqq. on EG, GF, *i.e.* double of the square on CD.

Then the sides of the triangle AEF represent the lines on which the three squares are to be constructed.

Is the angle AEF a right angle?

The angle AEF is made up of the angles AEC, FEG. Both of these are angles at the base of isosceles triangles, and both of the isosceles triangles are also right angled, therefore each of the angles must be equal to half a right angle (I. 32) and the two together, that is, angle AEF is a right angle.

Therefore the square on AF (sqq. on AD, DB) =
sqq. on AE, EF (double of sqq. on AC, CD I.47).

Considerations such as these probably suggested Euclid's construction given below, which differs slightly from the figure we have obtained.

SUMMARY OF PROPOSITION IX., THEOREM 9.—1ST METHOD.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Sqq. on AD, DB = twice sqq. on AC, CD.

Cons.—On AD describe the sq. ADFE.

In AE make AG = AC.

Through G and C draw ‖s to AC and AG meeting in H.

In EF, FD make FL, KF each equal to AC. Complete the square as above.

Proof.—Show that AH and MF = twice sq. on AC
and GL and CK = twice sq. on CD
MH = sq. on DB.

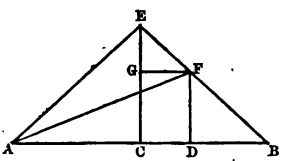
These four squares less MH = sq. on AD, that is sqq. on AD, DB = the four squares = twice sqq. on AC, CD.

EUCLID'S METHOD.

Cons.—Draw CE at rt. angles to AB, making it equal to AC.

Join EA, EB.

Draw DF, FG ‖ to CE and CD, join AF.



Proof.—ACE and BCE are isosceles and rt.-angled △s (*cons.*)

∴ each of the angles at their bases = half a rt. ∠.

∴ ∠ AEB (AEC + BEC) = a rt. ∠.

Again, in △s GFE, FDB the angles at the base are equal.

∵ one angle in each is a rt. ∠, and another half a rt. ∠.

∴ they are isosceles and GE = GF = CD, and DF = DB.

Then sqq. on AD, DB =
sq. on AF =
sqq. on AE, EF =
sqq. on AC, CE and sqq. on GF, GE
= twice sqq. on AC, CD. } (I. 47.)

SUMMARY OF PROPOSITION X., THEOREM 10.—1ST METHOD.

If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

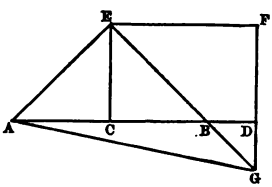
Sqq. on AD, DB = twice sqq. on AC, CD.

Cons.—Describe the sq. on AD (I. 46).

Construct two sqq. = sq. on CD as in II. 9.

Proof.—As in II. 9.

EUCLID'S METHOD.



Cons.—Draw CE at rt. angles to AB and = AC.

Join EA, EB.

Through E and D draw ‖s to CD and EC meeting in F.

Produce FD and EB to meet in G. Join AG.

Proof.—∠ AEB is a rt. ∠ (II. 9) and ∠ EBC = half a rt. ∠.

∴ DBG = half a rt. ∠ (I. 15) and BDG = a rt. ∠.

∴ DGB = half a rt. ∠ and BD = DG.

And sq. on AG = sqq. on AD, DG = sqq. on AD, DB (I. 47).

Also ∠ at F is a rt. ∠ and ∠ FGE = half a rt. ∠.

∴ ∠ FEG = half a rt. ∠ and FG = FE = CD (I. 34 and I. 6).

∴ sq. on EG = sqq. on EF, FG = twice sq. on CD.

And sq. on AE = twice the square on AC (II. 9).

∴ sqq. on AE, EG, that is, twice the sqq. on AC, CD = sq. on AG, that is, the sqq. on AD, DB.

This proposition concludes a group of ten theorems showing relations between the rectangles contained by straight lines divided in various ways.

Each theorem is an equation or statement of equality, and may be demonstrated by constructing the figure described on one side of the equation, and by means of parallel lines dividing the figure thus obtained into the rectangles and squares mentioned on the other side. Euclid adopts this method in the first eight propositions, but varies it, as we have seen, in the ninth and tenth. All of the theorems are susceptible of a double enunciation according as we consider the data to be one divided line, or the sum of two whole lines.

We find that several which require a separate enunciation on the supposition of one line being given may be grouped together if we suppose two lines to be given, thus reducing Euclid's ten propositions to four, with two corollaries.

The following table compares both methods of enunciation :—

II. 1. See opposite.	II. A.
II. 2.	If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.
If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.	Includes props. 2 and 3.
(Particular case of II. 1.)	
II. 3.	
If a straight line be divided into any two parts, the rect. contained by the whole and one of the parts = the rectangle contained by the two parts, together with the square on the aforesaid part.	
(Particular case of II. 1.)	

II. 4.

If a straight line be divided into any two parts, the square on the whole line = the squares on the parts, and twice the rectangle contained by the parts.

B.

The square on the sum of two lines = the squares on the two lines and twice the rectangle contained by those lines.

II. 5.

If a straight line be divided into two equal and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Rect. AD, DB and sq. on CD = sq. on CB.

C.

The rectangle contained by two straight lines, together with the square on half their difference, is equal to the square on half their sum.

Rect. AD, DB and sq. on CD = sq. on CB.

Includes Euclid's II. 5 and 6.

II. 6.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Rect. AD, DB and sq. on CB = sq. on CD.

II. 8.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

4 (Rect. AB, BC) with sq. on AC = sq. on sum of AB, BC.

C. Cor.

Four times the rectangle contained by two straight lines, with the square on their difference = the square on their sum.

4 (Rect. AB, BC) with sq. on AC = sq. on AB and BC.

Euclid's II. 8.

II. 7.

If a straight line be divided into any two parts, the squares on the whole line and one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Sqq. on AB, BC = twice rect. AB, BC and sq. on AC.

D.

The sum of the squares on two lines are equal to twice the rectangle contained by those lines, together with the square of the difference.

Sqq. on AB, BC = twice rect. AB, BC and sq. on BC.

Euclid's II. 7.

II. 9.

If a straight line be divided into two equal and also into two unequal parts, the squares on the unequal parts are together double of the squares on half the line, and on the line between the points of section.

Sqq. on AD, DB = twice sqq. on AC, CD.

Cor.

From Theorems B and D we infer that the squares on the sum and difference of two lines are double of the sum of the squares on the two lines.

Sqq. on AD, DB = twice sqq. on AC, CD.

Euclid's II. 9 and 10.

II. 10.

If a straight line be bisected and produced to any point, the square on the whole line thus produced, and that on the part produced, are double of the squares on half the line bisected, and on the line made up of the half and the part produced.

Sqq. on AD, DB = twice sqq. on AC, CD.

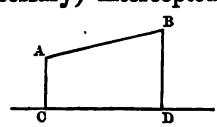
LECTURE XXI.

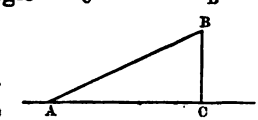
PROPOSITIONS 11–14.—ANALYSIS OF BOOK II.

THE second book concludes with two problems and two theorems depending on the principles laid down in the previous ten propositions. We will first consider the theorems.

Props. XII. and XIII. which complete the subject begun in I. 47, show the relation of the square subtending (1) an obtuse, (2) an acute angle, to the squares on the sides containing either of these angles.

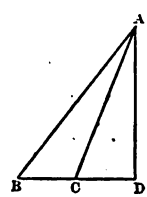
In the enunciation of these theorems the term projection is made use of. The projection of one line upon another is that part of the second (produced if necessary) intercepted between perpendiculars let fall from the extremities of the first. If, however, the lines meet at an angle, the projection is the line intercepted between that angle and the perpendicular let fall from the other extremity.





Prop. XII. shows that in obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing it by twice the rectangle contained by one of these sides and the projection of the other upon it.

Let ABC be a triangle having an obtuse angle ACB, from A let fall a perpendicular upon BC produced to D. We are required to show that sq. on AB = sqq. BC, CA, and twice the rectangle BC, CD. What do we know about the squares on these lines?



(1) Sq. on AB = sqq. AD, DB (I. 47).

(2) sq. on DB = sqq. on BC, CD and twice rectangle BC, CD (II. 4).

Then sq. on AB = sqq. on AD, CD, BC, and twice the rect. BC, CD.

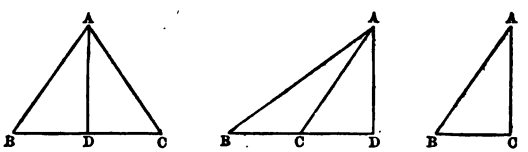
But sq. on AC = sqq. on AD, DC.

Therefore the sq. on AB = sqq. on AC, CB and twice the rect. BC, CD.—Q.E.D.

Prop. XIII. shows that in any triangle the square on the side subtending an acute angle is less than the squares on the sides containing it by twice the rectangle contained by either of those sides and the projection of the third side upon it.

Let ABC be any triangle of which ∠ ABC is an acute angle; let fall a perpendicular from A upon BC (produced if necessary). We are required to show that the square on AC with twice rect. CB, BD = the sqq. on CB, BA.

The perpendicular may fall upon BC produced, as in the figure of II. 12, or it may fall within the figure, or lastly, if the triangle is right-angled, it may coincide with a side of the triangle.



In the last case BC is itself the projection of AB upon BC, therefore the rectangle CB, BD is represented by the square on BC. Hence the demonstration of this case is obvious, for

Sq. on AB = sqq. on AC and BC.

therefore the sq. on AC is less then sqq. on AB, BC by twice the sq. on BC (that is rect. CBBC).

Let us examine the first case.

Sq. on AC = sqq. on AD, DC (I. 47).

Twice rect. CB, BD and sq. on CD = sqq. on BC and BD (II. 7).

Then the sq. on AC and twice the rect. CB, BD = sqq. on AD, BD, and BC.

But sq. on AB = sqq. on AD, BD.

Therefore the sq. on AC and twice the rect. CB, BD = sqq. AB and BC.

This proof also applies to the second case, for though BD in Case I. corresponds to BC in Case II., the rect. CB, BD is contained by 'the whole line and a part' in each.

Euclid considers only the first case and enunciates the theorem 'In acute-angled triangles, &c.'

The converse of both of these propositions is easily demonstrated by the indirect method (as in I. 19).

Summary of Proposition XII., Theorem 11.

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing it by twice the rectangle contained by one of these sides and the projection of the other upon it.

Sq. on AB = sqq. on AC, CB, and twice rect. BC, CD.

Cons.—Nil.

Proof.—Sq. on AB = sqq. on AD, BD (I. 47).

Sq. on BD = sq. on BC, CD, and twice rect. BC, CD (II. 4).

Sq. on AC = sqq. on AD, CD.

∴ sq. on AB = sqq. on AC, CB, and twice rect. BC, CD.—Q.E.D.

Summary of Proposition XIII., Theorem 12.

In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing it by twice the rectangle contained by either of those sides and the projection of the other upon it.

Sq. on AC and twice rect. CB, BD = sqq. on AB, BC.

Cons.—Nil.

Proof.—Cases I. & II. Sq. on AC = sqq. on AD, DC (I. 47).

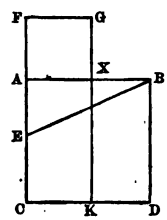
Twice rect. CB, BD and sq. on DC = sqq. on CB, BD (II. 7).

$\therefore$ sq. on AC and twice rect. CB, BD
= sqq. on CB, BD, DA.
= sqq. on CB, BA (I. 47).

Case III. is manifest (I. 47).

Prop. XI., the first problem in the book is 'To divide a straight line into two parts, so that the rectangle contained by the whole line and one of the parts is equal to the square on the other.'

Adopting the method of analysis we will assume that the point X in the line AB is the one required. On AX describe the square AFGX, and producing GX to K, making XK = AB, construct the rectangle XBDK.



These figures are equal in area (hyp.); also AB and BD form two sides of the square on AB. Complete this square. Then the square on AB = the rectangle FK; since FX and XD are equal, and AK is common to both figures.

Now the rectangle FK is contained by a line CA produced to F and the part produced (AF = FG).

Looking through the theorems on the segments of a straight line, we find that such a rectangle together with the square on half the original line = the square on the line made up of half the line and the part produced. Bisect AC in E, then rect. CFFA and sq. on AE = sq. on EF (II. 6).

Also, since the sq. on AB = rect. CFFA,

Sq. on AB and sq. on AE = sq. on EF.

Now AB and AE are at right angles to each other, therefore joining their extremities we obtain a right-angled triangle, and the square on EB the hypothenuse = the sqq. on AB, and AE = sq. on EF; it follows then that EB = EF.

And EF is made up of half AC = AB, which is given, and AF a side of the square which we have assumed. EB then forms a measure for EF and enables us to determine the distance to which CA is to be produced. We can now retrace

our steps and obtain a construction which solves the problem thus:—

On AB describe a square ABDC (I. 46).
Bisect AC in E. Join EB.
Produce EA to F, making EF = EB.
On AF describe sq. AFGH.
Produce GH to meet CD in K.

Sq. FH = rect. HD, and H is the point required.
The proof is obvious from the analysis.

SUMMARY OF PROPOSITION XI., PROBLEM 1.

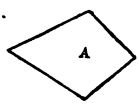
To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.

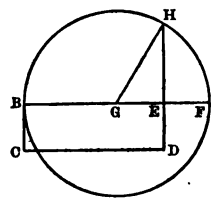
Cons.—On AB describe the square ACDB (I. 46).
Bisect AC in E and join EB.
Produce CA to F, making EF = EB.
On AF describe the square AFGH. Produce GH to K.
Proof.—Rect. CFFA and sq. on AE = sq. on EF (II. 6).
= sq. on EB = sqq. on EA, AB.
∴ rect. CFFA = sq. on AB.
Take from each the common part AK.
Then FH (sq. on AH) = HD (rect. AB, BH).
∴ AH is divided at H, &c.—Q.E.F.

Prop. XIV. completes the problem of the quadrature of a rectilineal area, in which I. 42, 44, and 45 have been successive stages. It shows how 'to describe a square that shall be equal to a given rectilineal figure.' Let A be the given rectilineal figure, then, taking up the problem at the point we left off in I. 45, we can construct a parallelogram equal to A, having its sides inclined at any angle we please. Let them be inclined at right angles, and describe the rectangle BCDE equal to A. Then if BC = CD the rectangle is a square, and the problem is solved. But if not, it remains to construct a square equal to the rectangle BCDE.

A square and a rectangle were constructed equal in the figure of the last problem. Can we adopt that, or a similar construction?

First, that the lines containing the rectangle may be represented by one straight line produce BE to F, making EF=ED. Then, looking through the theorems on the segments of a straight line, we find that if BF be bisected in G, the rectangle BE, EF (that is, the rectangle BCDE) with the square on GE = the square on GF (II. 5). In the last figure BE (= EF, the line corresponding to GF) is the hypothenuse of a right-angled triangle, of which the line corresponding to GE and a side of the required square contain the right angle. Can we construct such a triangle?





Since a line equal to GF is to be the hypothenuse, the vertex of the triangle must be in the circumference of a circle of which GF is a radius. Describe this circle, taking G as the centre.

Again, since GE is to be one of the sides containing the right angle, the vertex must be at some point in a line perpendicular to GE, that is in DE produced.

Thus either of the two points of intersection of DE produced with the circle will form the vertex of a triangle satisfying the required conditions.

Produce DE to meet the circumference of the circle in H and join GH.

Then HGE is a right-angled triangle and

sqq. on GE and EH = sq. on GH
= sq. on GF = rect. BCDE & sq. on GE.

Take away the common square and

sq. on EH = rect. BCDE
= given rectilineal figure A.

Summary of Proposition XIV., Problem 2.

To describe a square that shall be equal to a given rectilineal figure.

Cons.—Describe the rect. BCDE = A (I. 45).

Produce BE to F making EF = ED, bisect BF in G, with centre G and distance GF describe a semicircle.

Produce DE to meet it in H. Join GH.

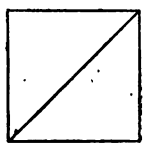
Proof.—Rect. BE, EF and sq. on GE = sq. on GF (II. 5)
= sq. on GH = sqq. on GE and EH

∴ rect. BE, EF alone = sq. on EH.

∴ rect. BE, EF (*i.e.* BCDE = A) = sq. on EH.

Q.E.F.

The second book may be divided into two parts. Part I. consists of ten theorems showing the relations between the rectangles contained by various segments of a straight line. There is an analogy between these and certain elementary principles of algebra and arithmetic, thus II. 4 is the geometrical proof of the algebraical formula $(a+b)^2 = a^2+2ab+b^2$. The whole of Part I. may be demonstrated algebraically, but these demonstrations are not so satisfactory as Euclid's methods, owing to the existence of 'incommensurable magnitudes.' Thus by geometry we can express the exact length of the diagonal of a square which can only be determined approximately by algebra or arithmetic, *e.g.* let the side of a square be 1, 2, or 3 inches, then the diagonal is $\sqrt{2}$ $\sqrt{8}$ $\sqrt{18}$ in. and each of these numbers consists of an integer and an infinite decimal.



Part II. contains:—

(*a*) Two theorems showing the relations between the squares on the sides of a triangle containing an acute or an obtuse angle, and the square on the side subtending it. These, with I. 47, are of great importance in trigonometry.

(*β*) Two problems, one showing how to divide a straight line into certain segments; the other, how to construct a

square equal to any irregular polygon. To this problem I. 42, 44, and 45 are stepping-stones. The demonstration of these four propositions depends on the principles learnt in Part I.

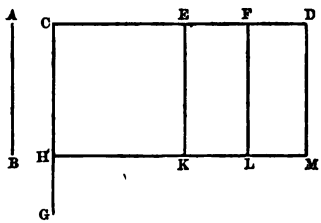
PROPOSITION A (EUCLID'S II. 1, 2, 3).

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

Let AB and CD be the two given lines, of which CD is divided at E and F:—

Rect. AB,CD = rectangles AB,CE, AB,EF, and AB,FD.

From C draw CG at right angles to CD, and make CH = AB (I. 12).



Through H draw HM ‖ to CD.

Through E, F, and D draw EK, FL, DM ‖ to CH.

Then rect. CM = rectangles CK, EL, FM.

But CM is contained by AB, DC, for CH = AB.

And CK is contained by AB, CE, for CH = AB.

And DL is contained by AB, EF, for EK = CH = AB.

In like manner FM is contained by AB, FD.

∴ rect. AB, CD = the rectangles AB, CE; AB, EF; AB, FD.

Wherefore if there be two straight lines, &c.—Q.E.D.

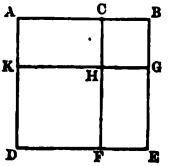
PROPOSITION B (EUCLID'S II. 4).

The square on the sum of two lines is equal to the squares on the two lines, and twice the rectangle contained by those lines.

Let AC, CB be the two given straight lines:

The square on AB their sum = the squares on AC, CB, and twice the rect. AC, CB.

On AB describe the sq. ADEB (I. 46).
Through C draw CF ‖ to AD,
In BE cut off BG = CB,
Through G draw GHK ‖ to AB,
Then AE = CG, KF, AH, and HE.



And AE = sq. on AB (cons.),
CG = sq. on CB, ∵ BG = CB and ∠ CBG is a rt. ∠.
KF = sq. on AC, ∵ AD and DE each = AB,
and AK and FE each = CB,
∴ the remainders KD, DF each = AC,
and angle at D is a rt. ∠.
AH = rect. AC, CB ∵ CH = CB,
HE = rect. AC, CB ∵ HF = AC and HG = CB.
∴ sq. on AB = sqq. on AC, CB, and twice rect. AC, CB.
Wherefore the square on the sum of two st. lines, &c.—
Q.E.D.

Cor. The square on the sum of two straight lines = four times the square of half their sum.

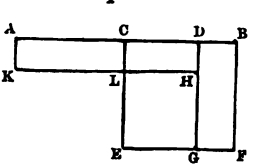
PROPOSITION C. (II. 5 and 6, and Cor. = II. 8).

The rectangle contained by two straight lines, together with the square on half their difference, is equal to the square on half their sum.

Let AD, DB be the two given lines, placed in one straight line (bisect AB in C, CB = half their sum, CD = half their difference).

The rect. AD,DB and the sq. on CD = the sq. on CB.

On CB describe the sq. CEFB (I. 46).



Through D draw DG ‖ to CE (I. 31).
From it cut off DH = DB (I. 3).
Through H draw HLK ‖ to AB,
and through A draw AK ‖ to DH (I. 31).
Then rect. DF = rect. AL ∵ BF = BC = AC and DB = DH = CL.

Also from the last proposition it is evident that LG = sq. on CD.

Then rects. AL and CH, that is, rect. AD,DB = rects. DF and CH.

To each of these add LG, that is, the sq. on CD.

Then rect. AD, DB, and the square on CD = CH, DF, and LG, which make up the sq. on CB.

Wherefore the rectangle contained by two straight lines, &c.—Q.E.D.

Cor. I.—From this construction it is manifest that the difference between the squares on any two straight lines is equal to the rectangle contained by the sum and difference of those lines.

Cor. II.—Four times the rectangle contained by any two lines together with the square on their difference = the square on their sum.

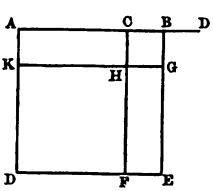
For square on the difference = four times sq. on half the difference, and square on sum = four times sq. on half the sum (II. B. Cor.).

Proposition D. (II. 7, 9, and 10).

The square on the difference of two straight lines, together with twice the rectangle contained by these lines, is equal to the sum of the squares on the two lines.

Let AB, BD be the two straight lines; make CB = BD, then AC is their difference, and

Sq. on AC and twice the rect. AB, BC = sqq. on AB, BC.



Draw the square on AB and construct a figure as in the previous props.

Then AE is sq. on AB,
KF is sq. on AC,
CG is sq. on BC or BD,
AG and CE each = rect. AB, BC.

Then twice the rect. AB, BC = AG and CE = gnomon AGF and sq. on BC.

To each of these add KF, the sq. on AC.

Then twice the rect. AB, BC and the sq. on AC = gnomon AGF and KF, and sq. on BC = sqq. on AB and CB.

Wherefore the square on the difference of two straight lines, &c.—Q.E.D.

Cor.—The squares on the sum and difference of two straight lines = twice the sum of the squares of the two lines.

For sq. on difference and twice rect. contained by the two lines = sum of sqq. on the two lines (II. D).

Also sq. on sum of two lines less twice the rect. contained by the lines = sum of sqq. on the two lines (II. B).

Whence we infer that—

The sqq. on the sum and difference = twice the sum of the squares on the two given lines.—Q.E.D.

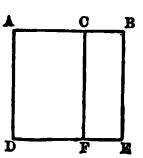
BOOK II.

Proposition I. (*See* Prop. A.)

Proposition II., Theorem 2.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts at C.



The rectangles AB, AC and AB, BC = the sq. on AB.

On AB describe the sq. ADEB (I. 46).

Through C draw CF ∥ to AD or BE.

Then AE = rects. AF, CE.

But AE = sq. on AB (cons.)

And AF = rect. AB, AC ∵ AD = AB.

And CE = rect. AB, BC ∵ BE = AB.

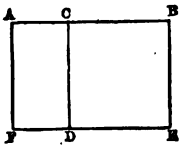
∴ rects. AB, AC, and AB, CB = sq. on AB.

Wherefore if a straight line, &c.—Q.E.D.

Proposition III., Theorem 3.

If a straight line be divided into any two parts the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let the straight line AB be divided into any two parts at C. The rect. AB, BC = the rect. AC, CB, and the sq. on CB.



On CB describe the sq. CDEB.

Through A draw AF ∥ to CD meeting ED produced in F.

Then AE = the rects. AD, CE.

But AE = rect. AB, BC ∵ BE = BC.

And AD = rect. AC, CB ∵ CD = CB.

And CE = sq. on CB (cons.)

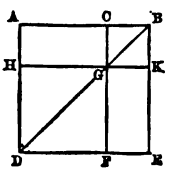
∴ rect. AB, BC = rect. AC, CB and sq. on CB.

Wherefore if a straight line, &c.—Q.E.D.

Proposition IV., Theorem 4.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

Let the straight line AB be divided into two parts at the pt. C; the sq. on AB = sqq. on AC, CB, and twice the rect. AC, CB.



On AB describe the square ADEB (I. 46).

Join BD.

Through C draw CGF ∥ to AD or BE.

Through G draw HK ∥ to AB or DE (I. 31).

Then, ∵ CF is ∥ to AD and BD meets them.

The extr. ∠ CGB = int. and opp. ∠ ADB.

Also ∵ BA = AD (cons.), ∠ ADB = ABD (I. 5).

∴ ∠ CGB = ∠ CBG (ax. 1).

And the side CG = side CB (I. 6).

But CB also = GK and CG = BK (I. 34).

$\therefore$ the figure GK is equilateral.

Again, $\because$ CG is $\parallel$ to BK and CB meets them; the $\angle$s KBC, GCB = 2 rt. $\angle$s (I. 29).

And KBC is a rt. $\angle$; $\therefore$ GCB is a rt. $\angle$.

And $\therefore$ also $\angle$s CGK, GKB opposite to them are rt. $\angle$s.

$\therefore$ the figure CGKB is also rectangular.

$\therefore$ it is a square and it is on the side CB.

For the same reason HF is also a square, and it is on HG = AC.

And HF, CK are the squares on AC, CB.

And AG is the rect. AC, CB, for CG = CB.

And the complement GE = AG.

$\therefore$ AG and GE = twice the rect. AC, CB.

And these four figures HF, CK, and AG, GE make up the whole figure ADEB, that is the square on AB.

$\therefore$ the sq. on AB = the sqq. on AC, CB and twice the rect. AC, CB.

Wherefore if a straight line, &c.—Q.E.D.

Cor. I.—The square on the whole line = four times the square on half the line.

Cor. II.—From this demonstration it is manifest that $\parallel^{\text{grams}}$ about the diameter of a square are also squares.

Proposition V., Theorem 5.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be divided into two equal parts at C, and into two unequal parts at D.

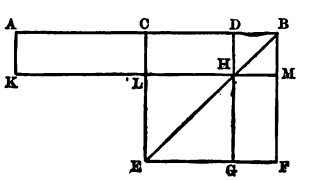
The rect. AD, DB and the sq. on CD = the square on CB.

On CB describe the square CEFB (I. 46).

Join BE.

Through D draw DHG $\parallel$ to CE or BF.

Through H draw KLM $\parallel$ to CB or EF.



Through A draw AK ‖ to CL or BM meeting KLM in K. (I. 3.)

Because the comps. CH and HF are equal, and DM is common to the rects. CM and DF, these rects. are equal.

And CM = AL, ∴ DF = AL.

To each of these add CH.

Then the gnomon CMG = AH, that is rect. ADDB ∵ DH = DB.

To each of these add LG, that is the sq. on CD (II. 4, Cor., and I. 34).

Then the gnomon CMG and LG, that is the sq. on CB = rect. AD, DB, and the sq. on CD.

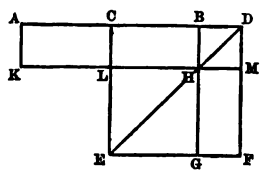
Wherefore if a straight line, &c.—Q.E.D.

Cor.—From this proposition it is manifest that the difference of the squares on two unequal straight lines AC, CD, is equal to the rectangle contained by their sum and difference.

Proposition VI., Theorem 6.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line, which is made up of the half and the part produced.

Let the straight line AB be bisected at C and produced to D. The rect. AD, DB and the sq. on CB = sq. on CD.



On CD describe the sq. CEFD (I. 46).

Join DE, and construct the figure as in the preceding proposition.

Then rect. AL = CH ∵ AC = CB, and the comp. CH = HF ∴ AL = HF.

To each of these add CM.

Then AM, that is the rect. AD, DB = the gnomon CMG.

To each of these add LG, that is the sq. on CB.

Then rect. AD, DB and the sq. on CB = the gnomon CMG, and LG = the sq. on CD.

Wherefore if a straight line, &c.—Q.E.D.

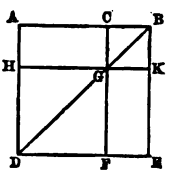
Proposition VII., Theorem 7.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts at the point C.

The sqq. on AB, BC = twice the rect. AB, BC and the sq. on AC.

On AB describe the square ADEB, and construct the figure as before.



Then the comp. AG = the comp. GE.

To each of these add CK.

Then CE = AK and the two rects. = twice the rect. AB, BC.

Also AK and CE = the gnomon AKF and the sq. on CB.

∴ gnomon AKF and the sq. on CB = twice the rect. AB, BC.

To each of these add HF, that is the sq. on AE.

Then the gnomon AKF and the sqq. on AC, CB = the sq. on AC and twice the rect. AB, BC.

That is the sqq. on AB and BC = twice the rect. AB, BC and the sq. on AC.

Wherefore if a straight line, &c.—Q.E.D.

Proposition VIII., Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

Let the straight line AB be divided into any two parts at C. Four times the rect. AB, BC and the sq. on AC = the sq. on the sum of AB and BC.

Produce AB to D making BD = CB.

On AD describe the sq. AEFD and construct two figures as before.

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Then ∵ CB = BD, and also = GK, and BD = KN.
∴ GK = KN.
For the same reason PR = RO.
Also rect. CK = rect. BN and rect. GR
= rect. RN (I. 36).
And CK = RN ∵ they are complements
of ||gram CO, also BN = GR.
∴ the four rects. are equal to one another, and are quadruple of one of them CK.
Similarly it may be shown that the four rects. AG, MP, PL, RF, are equal, and are quadruple of one of them AG.
∴ the eight rectangles which make up the gnomon AOH are quadruple of AK, that is, the rect. ABBC.
To each of these add XH, that is the sq. on AC.
∴ four times the rect. AB, BC and the sq. on AC
= the gnomon AOH and XH
= sq. on AD,
that is on sum of AB and BC.

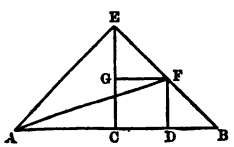
Wherefore if a straight line, &c.—Q.E.D.

Proposition IX., Theorem.

If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts at C and into two unequal parts at D.

The sqq. on AD, DB = twice the sqq. on AC, CD.



From C draw CE at rt. angles to AC (I. 11,) and make CE = AC (I. 3).

Join EA, EB.

Through D draw DF ‖ to CE, and through F draw FG ‖ to BA.

Join AF.

Then ∵ AC = CE, ∠ CAE = ∠ AEC.

And ∠ ACE is a rt. ∠, ∴ ∠s CAE and AEC are each half a rt. ∠ (I. 32).

For the same reason ∠s CEB, EBC are each half a rt. ∠.
∴ the whole ∠ AEB is a rt. ∠.

Again ∵ ∠ GEF is half a rt. ∠, and ∠ EGF is a rt. ∠ (I. 29).

∴ ∠ GFE is half a rt. ∠ and GE = GF (I. 6).

Also ∵ ∠ at B is half a rt. ∠, and ∠ FDB is a rt. ∠ (I. 29).

∴ ∠ DFB is half a rt. ∠ and DF = DB (I. 6).

And ∵ AC = CE, the sqq. on AC, CE = twice the sq. on AC.

But the sq. on AE = sqq. on AC, CE = twice the sq. on AC (I. 47).

Similarly the sq. on EF = sqq. on GF, FE = twice the sq. on GF = twice the sq. on CD (I. 34).

∴ the sqq. on AE, EF = twice the sqq. on AC, CD.

But the sq. on AF = the sqq. on AE, EF = twice the sq. on AC, CD (I. 47)

And the sq. on AF also = the sqq. on AD, DF (I. 47), that is the sqq. on AD, DB, ∵ DF = DB.

∴ the sqq. on AD, DB = the sq. on AF = twice the sqq. on AC, CD.

Wherefore if a straight line, &c.—Q.E.D.

Proposition X., Theorem.

If a straight line be bisected and produced to any point, the square on the whole line thus produced and the square on the part of it produced are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected in C and produced to D.

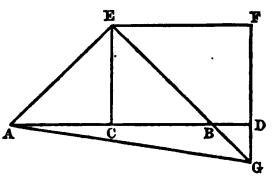
The sqq. on AD, DB = twice the sqq. on AC, CD.

From C draw CE at rt. angles to AC (I. 11).

And make CE = AC (I. 3).

Join EA, EB.

Through E draw EF ‖ to AB, and through D draw DF ‖ to CE (I. 31).



Then ∵ EF meets the ‖s EC, FD, the ∠s CEF, EFD = two rt ∠s.

∴ the ∠s BEF, EFD are together less than two rt. ∠s.

∴ The straight lines EB, FD will meet if produced towards B and D.

Let them meet at G and join AG.

Then ∵ AC = CE and ∠ ACE is a rt. ∠,

∠s CAE, EAC each = half a rt. ∠ (I. 5, I. 32).

Similarly ∠s CEB, EBC each = half a rt. ∠.

∴ The whole ∠ AEB = a rt. ∠.

And ∠ DBG = the vertical opposite ∠ EBC = half a rt. ∠ (I. 15).

And ∠ BDG = the alternate ∠ DCE = a rt. ∠ (I. 29).

∴ the remaining ∠ DGB = half a rt. ∠ = ∠ DBG.

∴ BD = DG (I. 6).

Again, ∵ ∠ EGF = half a rt. ∠.

And ∠ EFD = ∠ ECD = a rt. ∠ (I. 34).

∴ the remaining ∠ FEG = half a rt. ∠ = ∠EGF.

∴ GF = FE (I. 6).

And ∵ AC = CE, the sqq. on AC, CE = twice the sq. on AC.

But the sq. on AE = sqq. on AC, CE = twice the sq. on AC (I. 47).

Similarly the sq. on EG = sqq. on EF, FG = twice the sq. on EF = twice the sq. on CD (I. 34, I. 47).

∴ the sqq. on AE, EG = twice the sqq. on AC, CD.

But the sq. on AG = the sqq. on AE, EG = twice the sqq. on AC, CD.

And the sq. on AG also = the sqq. on AD, DG (I. 47),

that is, the sqq. on AD, DB ∵ DB = DG.

∴ the sqq. on AD, DB = twice the sqq. on AC, CD.

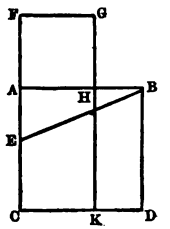
Wherefore if a straight line &c.—Q.E.D.

Proposition XI., Problem.

To divide a given straight line into two parts so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Let AB be the given straight line—

It is required to divide it into two parts, so that the rect. contained by the whole and one of the parts = the sq. on the other.



On AB describe the sq. ABDC.

Bisect AC at E, join BE.

Produce EA to F making EF = EB.

On AF describe the sq. AFGH.

AB shall be divided at H so that the rect. AB, BH = sq. on AH.

Produce GH to K.

Then the rect. CF, FA and the sq. on AE
= the sq. on EF (II. 6).
= the sq. on EB ∵ EB = EF.

And the sq. on EB = the sqq. on AB, AE (I. 47).

Take away the common sq. AE,

Then the rect. CF, FA, that is FK = the sq. on AB, that is AD.

Take away the common part AK.

The remainder FH, that is, the sq. on AH = HD, that is, the rect. AB, BH ∵ BD = AB.

Wherefore the straight line, &c.—Q.E.F.

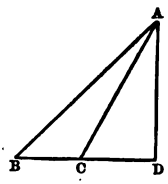
Proposition XII., Theorem.

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing it, by twice the rectangle contained by one of those sides and the projection of the other upon it.

Let ABC be a △, having an obtuse ∠ ACB; and let CD be the projection of AC on BC produced.

The sq. on AB = the sqq. on AC, CB and twice the rect. BC, CD.

For the sq. on BD = the sqq. on BC, CD and twice the rect. BC, CD.



To each of these add the sq. on DA.

∴ the sq. on BD, DA = the sqq. on BC, CD, DA and twice the rect. BC, CD.

But the sq. on AB = the sqq. on BD, DA (I. 47).

And the sq. on AC = the sqq. on CD, DA.

∴ the sq. on AB = the sqq. on AC, BC, and twice the rect. BC, CD.

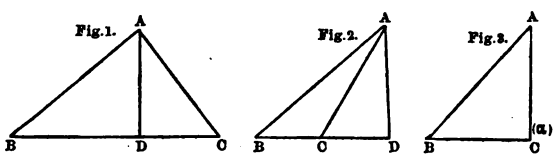
Wherefore in obtuse-angled triangles, &c.—Q.E.D.

Proposition XIII., Theorem.

In every triangle the square on the side subtending an acute angle is less than the squares on the sides containing it by twice the rectangle contained by either of these sides and the projection of the other upon it.

Let ABC be any Δ, having the ∠ ABC acute, and let DB be the projection of AB upon BC, or BC produced.

The sq. on AC and twice the rect. BC, BD = the sqq. on AB, BC.



For in the first case BC is divided into two parts at D.

In the second case BD is divided into two parts at C.

∴ in both, the sqq. on BC, BD = twice the rect. BC, BD and sq. on CD (II. 7).

Add to each the sq. on DA, then

The sqq. on BC, BD, DA = the sqq. on CD, DA, and twice the rect. BC, BD.

But sq. on BA = sqq. on BD, DA.

And sq. on AC = sqq. on CD, DA.

∴ the sq. on AC and twice the rect. BC, BD = the sqq. on AB, BC.

Case III. needs no demonstration.

Wherefore in any triangle, &c.—Q.E.D.

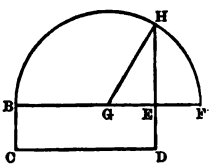
Proposition XIV., Problem.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure:

It is required to describe a square equal to A.

Describe the rectangular ||gram BCDE = A (I. 45).



Then if BE = ED, BCDE is a sq., and what was required is done.

If not, produce BE to F making EF = ED (I. 3).

Bisect BF in G.

From centre G at the distance GF describe a semicircle BHF.

Produce DE to H, join GH.

Then the rect. BE, EF and the sq. on GE
= the sq. on GF (II. 5).
= the sq. on GH ∵ GH = GF.

But the sq. on GH = the sqq. on GE, EH (I. 47).

∴ the rect. BE, EF and the sq. on GE = the sqq. on GE, EH,

And the rect. BE, EF alone = the sq. on EH.

But the rect. BE, EF = rect. BCDE = rectilineal fig. A.

∴ the sq. on EH = rectilineal fig. A.

Wherefore a square has been made, &c.—Q.E.F.

APPENDIX.

BEFORE attempting the exercises appended, it will be well to consider the various methods of proof adopted in the preceding lessons.

Theorems may be demonstrated by—

I. The indirect proof, with *reductio ad absurdum*. This is generally used in demonstrating converse propositions, and is not often required in exercises. Either no construction is required, as in I. 19, or it is very simple, and follows at once from the assumption that the theorem is not true, *e.g.* I. 6, 14, 26.

II. The method of superposition. This is fully described in Lecture IV. I. 8 is a simple example of its use.

III. The method of comparison. A very large number of theorems require the equality, or inequality, of certain lines and angles to be established. This can often be effected by comparing them with other lines and angles whose relative size we know, either by means of the angles or lines already existing in the figure, as in I. 15, or more often by constructing others, which serve as a medium of comparison between those whose equality, or inequality, we wish to show. I. 17, 18, and 21, are good examples of this method.

IV. The method of analysis. A clue to the more difficult constructions can be obtained by this method, for example I. 16. (See Lecture VI.)

Problems may be solved by the method of *loci*, as I. 22, or by analysis, as I. 9, &c.

A problem is said to be determinate when, with the data, it admits of one definite solution. Thus, in I. 22, only one triangle can be constructed satisfying the required conditions.

A problem is indeterminate when it admits of more than one

definite solution, *e.g.* if it be required to construct a triangle equal in area to a given triangle, we know that any triangle on an equal base and between the same parallels as the given triangle will satisfy the required conditions (I. 38).

Sometimes certain 'limitations' exist, within which alone the solution of a problem is possible, *e.g.* in I. 22, of the three given lines any two must be greater than the third.

EXERCISES.

BOOK I.

LECTURE III. (PROPOSITIONS 1–3).

1. On a given base describe an isosceles triangle, whose equal sides shall each be equal to a given straight line.

2. Draw figures illustrating the following cases of I. 2 :—

(α) When A falls in BC produced.

(β) When points A and B coincide.

(γ) When A falls within BC.

3. In the figure of I. 2, if a side of the equilateral triangle is equal to the given line, show that the radius of the larger circle is equal to the diameter of the smaller.

4. Draw a straight line three times as long as a given straight line.

5. The shorter diagonal being given, construct a rhombus whose sides shall each be equal to the given diagonal.

6. Show that the two triangles into which the longer diagonal divides a rhombus are isosceles.

7. Construct an isosceles triangle, each of whose equal sides is double of the base.

8. Produce the shorter of two given straight lines so that it may be equal to the longer.

9. Construct the figure of I. 2 by producing BD and AD through the vertex of the equilateral triangle, and prove.

10. In the fig. of I. 1 produce CA, CB to meet the circles in E and F. Join EF, and show that the triangle EFC is isosceles.

LECTURE IV. (PROPOSITIONS 4–6).

1. Demonstrate formally the corollary to I. 5.

2. In an isosceles triangle the straight line which bisects the vertical angle also bisects the base.

3. The straight lines bisecting the angles at the base of an isosceles triangle (having base BC) meet at a point O. Show that OBC is an isosceles triangle.

4. Demonstrate formally the corollary to I. 6.

5. Let ABC be an equilateral triangle having its sides bisected in DEF. Show that the straight lines joining these points with the opposite angles are equal.

6. In a triangle ABC the vertical angle A is half of the angle B, which is bisected by the line BD, meeting the opposite side in D. Show that the triangle BDA is isosceles.

7. In the fig. of I. 5 let O be the point at which BG and FC intersect. Show that the triangle OBC is isosceles, and that FO is equal to GO.

8. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

9. Distinguish between equal and coincident triangles.

10. If a triangle ABC be turned over about its side AB, show that the line joining the two positions of C is perpendicular to AB.

11. An equiangular quadrilateral which has two adjacent sides equal to each other is also equilateral.

12. Produce BA, CA, sides of an equilateral triangle, through A, making AD, AE equal to BA, CA, each to each. Join BE, DE, DC.

What do we know about the figure thus formed ; and about the triangles of which it is composed ?

13. Two sides of a triangle being produced, if the angles on the other side of the base be equal, the triangle is isosceles.

14. If the angles of one triangle be equal to the angles of another, each to each, are the triangles necessarily equal ? What is the force of the expression, 'each to each' ? Exhibit a case when the angles are respectively equal, but not each to each.

LECTURE V. (PROPOSITIONS 7–8).

1. In the figure of I. 7, let the angle ABD be a right angle, and the angle ABC obtuse ; demonstrate the proposition under these conditions.

2. The opposite sides of a quadrilateral ABCD are equal : show the diagonal bisects it.

3. If two triangles have their sides equal, each to each, the triangles are equal in area.

4. ABC and DBC being two triangles, having their three sides equal each to each on a common base, join AD and prove that the triangles ADB, ADC are equal in all respects.

5. If a side of an equilateral triangle be double of the side of another equilateral triangle, what proportion will the two triangles bear to each other?

6. In the figure of I. 5 join O, the point at which BG and FC intersect with A, and show that this line bisects the angle BAC.

7. Show that the straight lines joining the points of bisection of the sides of an equilateral triangle divide the figure into four equilateral triangles.

8. The opposite angles of a rhombus are equal.

9. The diagonals of a rhombus bisect one another and the angles through which they pass.

10. BAC, BDC are isosceles triangles on opposite sides of a common base BC, join AD, and show that it bisects (α) the base, (β) the vertical angles of the triangles, (γ) the whole figure.

LECTURE VI. (PROPOSITIONS 9–12).

1. In the figure of I. 9, construct the equilateral triangle on the same side as A; demonstrate the case in which the point F falls without A.

2. Produce a given straight line so that the part produced may be one-fifth of the line when produced.

3. In a given straight line find a point which is equidistant from two given points.

4. If two isosceles triangles be described on the same base, the straight line joining their vertices (produced if necessary) bisects the base at right angles.

5. Divide an obtuse angle into four equal parts.

6. On a given base construct an isosceles triangle whose altitude is equal to the base.[1]

7. With common radius AB, draw two circles intersecting as in the fig. of I. 1, join the points of intersection: it is required to show that these straight lines bisect, and are perpendicular to each other.

[1] The altitude of a triangle is the perpendicular let fall from the vertex to the base (produced if necessary).

8. Find the locus of a point distant one inch from a given point P.

9. Find the locus of a point which is equally distant from two given points.

10. Find the locus of a point at a given distance from the circumference of a given circle.

11. What is meant by the intersection of loci? Find the position of a point equally distant from three given points, not in the same straight line.

LECTURE VII. (PROPOSITIONS 13–15).

1. The bisectors of an angle and its supplement are at right angles to one another.

(When two angles together make up two right angles, each is called the supplement of the other. Similarly, when two angles together make up one right angle, each is called the complement of the other.)

2. Show that the angles formed at the point of intersection of the diagonals of any quadrilateral are together equal to four right angles.

3. In the figure of I. 15, show that the bisectors of either pair of vertical angles are in the same straight line.

4. The straight lines AB, CD bisect each other in E; join their extremities, and show that the triangles ACD, BCD are equal in all respects.

5. Four radii of a circle make the opposite angles at the centre equal: show that these radii form two diameters of the circle.

6. All the angles made at the same point on the same side of a straight line are together equal to two right angles.

7. AB, BC, CD, DA, are the sides of a square taken in order, in AB take any point E and in BC, CD, DA, respectively make BF, CG, DH each equal to AE, show that EFGH is a square.

LECTURE VIII. (PROPOSITIONS 16–19).

1. ABC is any triangle, and P is any point within it. Join PA and PB. Show that the angle APB is greater than the angle ACB.

2. Any straight line drawn from one angle of an equilateral triangle to a point in the base is less than a side of the triangle.

3. The perpendicular is the shortest straight line which can be drawn from a given point to a given straight line; and of others that which is nearer to the perpendicular is less than the more remote.

4. Only two equal straight lines can be drawn from a given point to a given straight line.

5. What triangles can be bisected so that the parts coincide? In what case is this impossible?

6. In an obtuse-angled triangle the greatest side subtends the obtuse angle.

7. Demonstrate I. 17 by joining the vertical angle of the triangle with any point in the base.

8. Let AD be the longest and BC the shortest side of a trapezium, show that the angle ABC is greater than the angle ADC.

9. Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either if they be equal.

10. Two right-angled triangles are equal in all respects when the hypothenuse and a side of the one are equal to the hypothenuse and a side of the other (ex. 3).

Lecture IX. (Propositions 20–21).

1. The four sides of any quadrilateral are together greater than the sum of its two diagonals.

2. If the diagonals of a pentagon be drawn, they will form by their intersection another pentagon: prove that the sum of the perimeters of the two pentagons is less than the sum of the diagonals of the original pentagon.

3. If a point P be taken inside a quadrilateral, prove that the sum of the distances of P from the angular points is the least possible when P is situated at the intersection of the diagonals.

4. In any triangle the sum of the sides containing the vertical angle is greater than twice the line joining the vertical angle with the point of bisection of the base.

5. Four points lie in a plane, no one being within the triangle formed by joining the other three, determine the point the sum of whose distances from these four points is the least possible.

6. Show that any side of a triangle is less than half the sum of all three sides of the same triangle.

7. P is any point within the quadrilateral ABCD; show that

the sum of the distances of P from the angles at A, B, C, D, is greater than half the perimeter of the figure.

8. BDEC is a quadrilateral falling entirely within the triangle ABC, and upon a common base BC, show that the three remaining sides of the quadrilateral are together less than those containing the vertical angle of the triangle.

Lecture X. (Propositions 22–23).

1. Construct an isosceles triangle, having given the base and altitude.

2. Construct a triangle, having given two angles and the side between them. When is the problem impossible?

3. Construct a triangle, having given two sides and the angle between them.

4. ABC is a triangle, such that the angle at A is equal to the sum of the angles at B and C. Through the point C draw a line CD dividing ABC into two isosceles triangles.

5. Construct a triangle having given an angle, the side adjacent to it, and the sum of the other sides.

Lecture XI. (Propositions 24–26).

1. If, in an isosceles triangle, a straight line be drawn from the vertex perpendicular to the base, prove that the base and also the vertical angle are bisected.

2. The lines which bisect the angles of a triangle meet at a point within it.

3. Show that the perpendiculars let fall on two sides of a triangle from any point in the bisector of the angle between them are equal.

4. Through a given point draw a straight line such that the perpendiculars upon it from two given points may be on opposite sides of it and equal to each other.

5. In the second case of I. 26, show that if the triangle ABC be applied to the triangle DEF, the triangles must coincide (I. 16).

Lecture XII. (Propositions 27–29).

1. ABCD is a parallelogram having its side AD double of AB, the side AB is produced both ways to E and F till each produced

part is equal to AB, lines are drawn from C and D to E and F so as to cross within the figure, show that they meet at right angles. (The three interior angles of any triangle are together equal to two right angles.)

2. Show that any straight line drawn parallel to one of the sides of an equilateral triangle makes equal angles with the sides.

3. Assuming Playfair's postulate, demonstrate Euclid's Post. 6.

4. Show that Post. 6 is the converse of I. 17.

5. If through a point, equidistant from two parallel straight lines, two straight lines be drawn cutting them, show that they will intercept equal portions of those lines.

6. From a given isosceles triangle to cut off a trapezium with the same base as the triangle and the remaining three sides all equal to one another.

LECTURE XIII. (PROPOSITIONS 30–32).

1. Straight lines which are perpendicular to the same straight line are parallel to each other.

2. At what angles are the sides of a regular duodecagon, and of a pentagon, inclined to each other.

3. ABC is an equilateral triangle, BC is produced to D making CD equal BC, and AB is produced to E making BE double of BC, then AD and DE are joined. Determine the value of each angle of the triangle ADE.

4. To how many right angles are the interior angles of a hexagon equal?

5. Show that only three regular figures can be made to fill up completely the space round a point. Which are these?

6. Describe a triangle such that each of the angles at the base may be double of the third angle.

7. D is a point on the side BC of a triangle ABC, such that the angles ADC, ADB are respectively double of the angles ABC, ACB: show that the triangle ABC is right-angled, and that D bisects BC.

8. Trisect a given straight line.

9. What regular figures will fit together so as to cover a plane surface?

10. What pairs of regular figures on the same base will exactly cover a plane surface?

11. Draw patterns illustrating the two previous questions.

12. Construct the triangle, being given the vertical angle, the base, and the sum of the other sides.

13. If from the greater side AB of a triangle ABC a part AD be cut off equal to the shortest side AC, prove that the angle ACD = half the sum and the angle BCD = half the difference of the angles at the base.

14. Construct a parallelogram the adjacent sides of which shall be equal to two given lines, and the angle contained by them equal to a given angle.

Lecture XIV. (Propositions 33–34).

1. Prove that the straight lines which join the extremities of two equal and parallel straight lines towards opposite parts bisect each other.

2. The sum of the perpendiculars from an interior point in any parallelogram on the four sides is constant.

3. A rhombus is a parallelogram, and its diagonals bisect each other at right angles.

4. If one of the angles of a parallelogram be a right angle, does this determine all the other angles?

5. Through each angular point of a given triangle a straight line is drawn parallel to the opposite side. Show that the angles of the triangle formed by the three straight lines so drawn are equal to the angles of the original triangle, its sides respectively double of the sides, and its area four times the area of the same triangle.

6. If two parallel straight lines be intersected by two other parallel straight lines, show that the parts of the latter two intercepted between the former two are equal to each other.

7. If two opposite sides of a parallelogram be bisected, and lines be drawn from the points of bisection to the opposite angles, these lines trisect the diagonal.

8. The diagonals of a parallelogram bisect each other.

9. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.

10. In what quadrilateral figures are the diagonals equal?

11. In what quadrilateral figures do the diagonals intersect at right angles?

LECTURE XV. (PROPOSITIONS 35–41).

1. Any point P is taken in the line joining an angular point A of a triangle to the middle point of the opposite side BC. Prove that the triangles APB, APC are equal.

2. ABC is a triangle, BD bisects AC in D, CE bisects AB in E, BD and CE intersect in F. Prove that the quadrilateral ADFE is equal to the triangle FBC.

3. If in the base of a triangle ABC there be taken any two points P and Q, equidistant from the extremities of the base, and if through each of these points two lines be drawn parallel to the sides AB, AC, so as to form two parallelograms having PA and QA for diagonals, show that these parallelograms are equal.

4. Two triangles of equal area are on the same base and opposite sides of it. Prove that the straight line joining the vertices is bisected by the base, or the base produced, and conversely if the straight line be bisected, the triangles are of equal area.

5. Bisect a triangle by a line drawn from a given point in one of the sides.

6. Bisect a parallelogram by a line drawn from a given point, (α) within, or (β) without it.

7. Every triangle is the half of a rectangle having the same base and altitude.

8. Show that the four triangles into which the diagonals divide a parallelogram are equivalent.

9. If one side of a triangle be bisected and a straight line be drawn from the point of bisection parallel to the base, the third side is bisected by it, and the triangle cut off by it is one-fourth of the given triangle.

10. Find the locus of the vertices of equivalent triangles upon the same base, and on the same side of it.

LECTURE XVI. (PROPOSITIONS 42–45).

1. Describe a rhombus whose adjacent sides shall be to each other as 1 to 4.

2. State the process by which the area of any polygon may be converted into an equivalent rectangle.

3. Construct a rhombus equal to a given parallelogram.

4. Can a triangle be equal to a rectangle? If so, draw a rectangle equal to a given triangle.

5. Describe a triangle equal to a given rectangle.

6. Construct a triangle equal to any given quadrilateral figure.

7. One side of a rectangle being given, find the adjacent side, so that the rectangle may be equal to a given square.

Lecture XVII. (Propositions 46–48).

1. The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

2. The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

3. Demonstrate the converse of each of the preceding theorems.

4. The triangles ABC, DEF have the angles ABC, DEF right angles, and the sides AB, AC equal to DE, DF, each to each. Prove that the triangles are equal in every respect.

5. Being given the diagonal, construct the square.

6. Construct a square equal to the sum of two given squares.

7. Construct a square equal to the difference of two given squares.

8. Construct a square equal to three, four, or any number of squares.

9. Show how the squares on the sides of an isosceles right-angled triangle can be made to cover the square on the hypotenuse.

10. The parallelograms described on any two sides of a triangle are together equal to the parallelogram on its base having its side equal to and parallel to the line drawn from the vertex to the intersection of the produced exterior sides of the former figures.

Problems on the Construction of Triangles.

1. Construct an isosceles triangle having the angles at the base together equal to half the vertical angle.

2. Construct an isosceles triangle having the angles at the base equal to thrice the vertical angle.

3. Construct a right-angled triangle having given the hypotenuse and the difference of the sides.

4. Construct a right-angled triangle having given the hypotenuse and the sum of the sides.

5. Construct a right-angled triangle having given the perimeter and an angle.

6. Construct a right-angled triangle having given the hypotenuse and the perpendicular from the right angle upon it.

7. Construct a triangle having given the perimeter and each of the angles.

8. Construct a triangle equivalent to a given triangle having its base in the same straight line as that of the given triangle, and its vertex in a given line parallel to the base.

BOOK II.

LECTURES XVIII., XIX., XX. (PROPOSITIONS 1–10).

1. Show, by diagram, that $A(B + C) = AB + AC$.

2. ABC is a triangle right-angled at C, D and E are points in AB and AB produced such that $BD = BE = BC$. Show that the rectangle contained by AD and AE is equal to the square on AC.

3. Each of the sides of a rectangle is double of the corresponding side of another rectangle. How many times does the larger rectangle contain the other?

4. Any rectangle is the half of the rectangle contained by the diameters of the squares on two adjacent sides.

5. If a straight line be divided into two equal and two unequal parts, the squares of the two unequal parts are equal to twice the rectangle contained by those parts, together with four times the square of the line between the points of section.

6. The square of the difference of two straight lines is less than the sum of the squares on the two lines by twice the rectangle contained by those lines.

7. The difference between the squares on any two straight lines is equal to the rectangle contained by the sum and difference of those lines.

8. AB is divided into any two parts in C, and AC and BC are bisected in D and E. Show that the square on AE and three times the square on BE are equal to the square on BD, and three times the square on AD.

9. The square on the sum of two straight lines, together with

the square on their difference, is equal to twice the squares on the two straight lines.

10. The square of any straight line drawn from the vertex of an isosceles triangle to the base is less than the square of a side of the triangle by the rectangle contained by the segments of the base.

LECTURE XXI. (PROPOSITIONS 11–14).

1. If one angle of a triangle is $\frac{4}{3}$ of a right angle, the square of the side subtending it is equal to the sum of the squares of the sides containing it, together with the rectangle contained by those sides.

2. Produce a given line so that the rectangle contained by the whole line thus produced and the part produced shall be equal to a given square.

3. Straight lines which are equal and parallel have equal projections on any other straight lines.

4. Parallel straight lines which have equal projections on another straight line are equal; and equal straight lines which have equal projections on another straight line are equally inclined to that line.

5. Show that in every parallelogram the squares of the diagonals are together equal to the sum of the squares of all the sides.

6. Produce a line so that the sum of the squares on the whole line thus produced and on the part produced may be equal to three times the square on the given line.

7. If ABC is a triangle having the angles at B and C double of the angle at A, show that the square on AB = the square on BC, together with the rectangle contained by AC, CB.

8. Describe a rectangle equal to a given square, and having the sum of two of its adjacent sides equal to a given straight line.

9. Find a square equal to a given equilateral triangle.

10. The sum of the squares on any two sides of a triangle is equal to twice the sum of the squares on half the base, and on the line joining the vertical angle with the point of bisection of the base.

QUESTIONS SELECTED FROM THE EXAMINATION PAPERS OF THE UNIVERSITY OF LONDON, 1860–1882.

1. The longest side of a given quadrilateral is opposite the shortest; prove that either of the angles adjacent to the shortest side is greater than the opposite angle of the quadrilateral.

2. Show how to determine a line, cut as in the proposition (II. 11), when the greater segment is given.

3. Find a point within a triangle from which lines drawn to the angular points will divide the triangle into three equal parts.

4. The angle C of a triangle ABC is a right angle, CD is the perpendicular on AB, and E is the middle point of AB; prove that the squares on AE, ED, DC, are together equal to half the square on AB.

5. Describe an isosceles triangle having each of the angles at the base double of the third angle.

6. Can triangles be equal which cannot be made to coincide? Can squares or rectangles? Illustrate your answer by figures.

7. Show how to make a square equal to half a given square.

8. If a straight line be divided into two parts, the difference of the squares of the two parts is equal to the difference between the square on the whole line, and twice the rectangle contained by the whole and the lesser part.

9. Describe a square that shall be equal to the sum of two or more rectilineal figures.

10. Distinguish between a postulate and an axiom, and explain the necessity of a set of postulates and axioms as the basis of geometrical reasoning.

11. ABC is an isosceles triangle and in BA and CA produced points F and G are taken such that AF and AG are equal; prove that the angles AFC and AGB are equal.

12. Determine as a fraction of a right angle the interior angle of a regular octagon.

13. The difference between any two sides of a triangle is less than the third side.

14. Describe an equilateral triangle having its side equal to a given finite straight line, and having one of its angular points at a given point.

15. Describe the mode of proof called the *Reductio ad absurdum* and mention any instances of its use in the First Book of Euclid.

16. In a triangle ABC find a point P in the side AB such that if the parallelogram PBC be completed it shall be equal to the triangle ABC.

17. If isosceles right-angled triangles be described on the sides of a right-angled triangle as bases, prove that the triangle on the side subtending the right angle is equal to the triangles on the sides containing the right angle.

18. In which of Euclid's subsequent propositions is I. 6 first required?

19. Explain why Euclid's, or some equivalent postulate, is necessary, in addition to the definition of parallel straight lines, in order to conduct reasonings on the properties of parallel lines.

20. If any point P be joined to A, B, C, D, the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD.

21. Show that the diameter of the circle employed in the fig. of II. 14 is half the perimeter of the rectangle employed.

22. AB is parallel to CD and unequal to it, and they are joined towards the same parts by the straight lines AC and BD. If AC is equal to BD, show that AD is equal to BC.

23. A straight line is divided into two parts. Show that the sum of the squares on the parts is least when the straight line is bisected.

24. ABCD is a quadrilateral, AC and BD being the diagonals. The triangle ADC is half the triangle ABC, and the triangle BCD is half the triangle BAD. Prove that two of the sides of the quadrilateral are parallel, and that one of these is double of the other.

25. Show how to divide a right angle into six equal angles.

26. ABC is a triangle. Find a point D on the side BC such that the perpendicular from D on AC may be equal to DB.

27. ABC is a triangle having a right angle at C. Find a point

D in BC so that the difference of the squares on AB and AD may be equal to three times the difference of the squares on AD and AC.

28. AB, AC are two straight lines given in position. It is required to find in them two points, P and Q, such that the angle APQ may be equal to a given angle, and the difference of AP and PQ equal to a given length.

29. From a given point P without a given straight line AB draw a straight line making a given angle with AB.

30. OA, OB are two given straight lines meeting in O; and P is a given point in the angle AOB. Show how to draw a straight line through P so that the part intercepted between OA, OB may be bisected in P.

31. Divide a given straight line AB into two parts AC, BC, so that the rectangle contained by the whole AB and one of the parts BC may be equal to the square on AC.

Show that the difference of the squares on AC, BC is equal to the rectangle contained by AC, BC.

32. From a point within a quadrilateral straight lines are drawn to the four angular points. Determine the position of the point so that the sum of the four straight lines may have its least value.

33. Divide a given quadrilateral into two parts of equal area by a straight line drawn through an angular point.

34. Draw a line DE parallel to the base BC of a triangle ABC, cutting AB in D and AC in E, so that DE shall be equal to the sum of BD and CE.

35. Show how to construct a rectangle which shall be equal to a given square (1) when the sum, (2) when the difference, of two adjacent sides is given.

36. Show that the perpendicular is the shortest straight line which can be drawn from a given point to a given line.

37. AB, CD are two parallel straight lines, and AD, BC intersect in the point O. Prove that if AO is equal to OD, BO is equal to OC.

38. The base BC of an equilateral triangle is produced to D, so that the produced part CD is equal to BC. Show that the square of AD is equal to three times the square of BC.

39. On the sides of an equilateral triangle ABC, equilateral triangles BCD, CAE, ABF are described, all external. Show that the figure DEF is another equilateral triangle.

40. If a polygon of 40 sides be constructed, all of whose angles

are equal to each other, prove that each angle differs from two right angles by the tenth of a right angle.

41. ABC is a triangle, and CD is drawn from the angular point C perpendicular to the opposite side AB. If E be any point in CD, prove that the sum of the squares described on AC and EB is equal to the sum of the squares described on BC and AE.

42. If two straight lines AB and CD bisect one another at their point of intersection, prove that the figure ABCD is a parallelogram.

43. Of all rectangles of equal perimeter the square is the greatest.

44. Produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced shall be equal to the square of the given line.

45. If AEB, CED are two straight lines cutting one another at E, and if EF bisect the angle AEC, and EG bisect the angle CEB, show that EF is at right angles to EG.

46. If the diameters of a parallelogram are at right angles to one another, the parallelogram is either a square or a rhombus.

47. Find a straight line of which the square shall be equal to half the square of a given line.

48. Show that upon any given straight line as base it is possible to construct an equilateral triangle. Can only one such triangle be constructed? and, if more than one, how does that appear on the face of your construction?

49. If each interior angle of a polygon contains 165 degrees, how many sides has the polygon?

50. Describe a square equal to a given parallelogram (1) if the parallelogram is rectangular, (2) if it is oblique.

51. What is meant by saying that a right angle is an angle of 90 degrees?

52. If the three sides of one triangle are respectively equal to the three sides of another, prove that the three angles of the first are respectively equal to the three angles of the second triangle.

53. Specify the figures whose properties are discussed in the 'First Book of Euclid.' What is the least number of triangles into which a rectilineal plane figure of any given number of sides (*e.g.* 5 or 7 sides) can be dissected?

54. Deduce from I. 32 that all the exterior angles of any polygon are together equal to four right angles. Understanding by a regular polygon one whose sides and angles are all equal, find

the magnitude of each interior angle of a regular polygon of 72 sides.

55. AC is the longest side of the triangle ABC. Find in AC a point D such that the angle ADB shall be equal to twice the angle ACB.

56. How would you show that two circles which have equal radii are equal?

57. Given two points A and B on opposite sides of a given unlimited straight line, but at unequal distances from it, find a point P in this straight line such that the difference between AP and BP may be the greatest possible.

58. Show that the rectangle contained by two lines is equal to the excess of the square of half their sum above the square of half their difference.

59. What different kinds of magnitude are considered in plane geometry?

60. If the two diameters of a parallelogram are equal to one another show that the parallelogram is right-angled.

61. If the straight line BD meet the straight line ABC in B, show that the lines bisecting the angles ABD, CBD are at right angles to one another.

62. Upon AB, BC, CA, sides of the triangle ABC, perpendiculars are drawn from a point D, meeting the side or the sides produced on E, F, G respectively. Prove that $AE^2 + BF^2 + CG^2 = BE^2 + CF^2 + AG^2$.

63. ABC is a triangle of which the greatest angle is the angle at A. Prove that if a point D be taken in AB and a point E in AC, DE is less than BC.

64. If the sides BC, CA, AB of a right-angled triangle ABC are bisected in D, E, F, show that twice the squares of AD, BE and CF are together equal to three times the square on the hypotenuse.

65. If two given straight lines intersect, and a point be taken equally distant from each of them, prove that it lies on one or other of the two straight lines which bisect the angles made with one another by the given straight lines.

66. What fraction of a right angle is the exterior angle of a regular heptagon?

67. 'If a straight line falling on two other straight lines make the alternate angles equal to each other, these two straight lines shall be parallel.' What supposition is tacitly assumed concerning

the 'two straight lines' in this enunciation? Why is the proposition not necessarily true as it stands?

68. Investigate the sum of the exterior angles of any plane convex polygon.

69. State the relation which connects the obtuse angle of an obtuse-angled triangle, the sides containing the obtuse angle, and the segment lying in either one of these sides produced, and intercepted between the perpendicular drawn from the opposite acute angle upon the side so produced, and the obtuse angle.

70. Let ABC be any obtuse-angled triangle, C the obtuse angle, AD, BE perpendiculars upon BC, AC produced and meeting these produced lines in D and E. Prove that the rectangle of BC, CD is equal to that of AC, CE.

71. Bisect the angle between two unlimited straight lines which intersect.

Through a given point draw a straight line equally inclined to two given straight lines. How many such lines may be drawn?

72. Prove that the angle at the vertex of a triangle is less than, equal to, or greater than a right angle according as the line joining the vertex to the middle point of the base is greater than, equal to, or less than half the base.

73. The hypotenuse of three isosceles right-angled triangles form a right-angled triangle. Construct the figure, and prove that one of the given triangles is equal to the sum of the other two.

74. AOB, COD are two indefinite straight lines intersecting each other in the point O, and P is a given point in the plane of these lines. It is required to draw through the point P a straight line PXY, cutting AB in X and CD in Y, in such a manner that OX may be equal to OY.

Can this problem be solved in more than one way?

75. Construct an isosceles triangle whose base and angle at the vertex are given.

76. If through the middle point of a side of a triangle a straight line be drawn parallel to the base it will bisect the other side, and that part of it which is intercepted between the sides of the triangle will be equal to half the base.

77. If A be the point of intersection of the diagonals of a parallelogram, then every straight line drawn through A will meet opposite sides of the parallelogram in points equidistant from A, and will divide the $\|^{gram}$ into two equal parts.

78. In a triangle of any magnitude show that if two sides be equal the opposite angles are equal, and conversely.

79. Two sides of a triangle being supposed given in magnitude, show that the greater the angle between them the greater the third side, and conversely.

80. Through a given point on either side of a given indefinite right line draw the perpendicular and the parallel to the line.

81. A quadrilateral being formed by joining the middle points of adjacent sides of a rectangle, investigate what kind of quadrilateral it is.

82. On a given base construct a rectangle equal in area to a given parallelogram.

83. Prove that a quadrilateral which has two opposite sides parallel is equal in area to a rectangle between the same parallels on a base equal to half the sum of the parallel sides of the given quadrilateral.

84. Assuming that the diagonals of a parallelogram bisect each other, show how to construct a parallelogram which shall have its diagonals equal to two given lines, and one of the angles between them equal to a given angle.

85. Prove that the difference of the areas of two squares is equal to the area of a rectangle contained by two straight lines equal respectively to the sum and difference of the sides of the two squares.

86. Show how to construct a triangle having its vertex at a given point, its base on a given indefinite right line, and its base angles equal to two given angles.

87. Show that every two parallels to the sides of a parallelogram which intersect on either of its diagonals determine, with the two angles subtending the diagonal, two parallelograms of equal area.

88. By aid of the preceding show how to construct, on a given base, a parallelogram which shall be at once equiangular and equiareal with a given parallelogram.

89. Show, by a diagram, that the square of the sum exceeds the square of the difference of any two finite right lines by four times the rectangle contained by the lines.

90. By aid of the preceding, or otherwise, construct a rectangle which shall have at once a given area, and also either a given sum or a given difference of sides.

91. A parallel to the base BC of a triangle ABC cuts the sides,

produced if necessary, at D and E. Prove that the new triangle ADE has its angles equal to those of the original triangle ABC.

92. Construct an isosceles triangle which shall have its base equal to a given straight line, and the angle at its vertex equal to a given angle.

93. A straight line terminated by two parallel straight lines is bisected at a point C. Prove that every straight line through C which is terminated by the parallels is bisected at C.

94. Prove that in every isosceles triangle the bisector of the angle between the equal sides bisects the third side, and is perpendicular to it.

95. Prove that a triangle is isosceles if the bisector of any angle is perpendicular to the opposite side.

96. Two of the angles in a triangle which is not given are known to be equal to two given acute angles. Show how to make, with a given straight line, an angle which shall be equal to the third angle in the triangle.

97. Prove that no polygon of the ordinary form can have more than three of its exterior angles obtuse, or more than three of its interior angles acute.

98. Construct a parallelogram which shall have two adjacent sides and the diagonal through their point of intersection equal to three given straight lines of finite lengths.

99. Prove that in any right-angled triangle the rectangle contained by the hypotenuse and either part into which it is divided by the perpendicular from the right angle is equal in area to the square on the side adjacent to that part.

100. If a straight line be divided into two unequal parts, show, by a diagram, that half the sum of the squares of these parts is equal to the square on half the sum of the parts, together with the square on half their difference.

101. Divide a finite right line of given length into two parts, the sum of whose squares shall have a given value not less than twice the square of half the line.

102. Show that every parallel to the base of an isosceles triangle cuts off equal lengths on the sides measured from the vertex of the triangle.

103. Of the two diagonals of a parallelogram which is not a rectangle the greater connects the two acute, and the lesser the two obtuse, angles of the figure.

104. A rectilineal segment of any length being supposed divided

equally and unequally, show by any method that the rectangle under the equal parts exceeds the rectangle under the unequal parts by the square of the interval between the points of section.

105. Assuming the preceding property or otherwise show how to divide a rectilinear segment of any length into two parts, the rectangle under which shall be equal to three-fourths of the square of half the segment.

106. A number of equal triangles stand on the same base and on the same side of it. Prove that their vertices are all situated on the same right line which is parallel to their common base.

107. Divide a given right line into two parts so that the difference between the squares described on the parts shall be equal to a given rectangle.

108. Divide a given right line into two parts so that the rectangle under the parts shall be equal to a given rectangle; and point out when the solution becomes impossible.

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